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Local structure of Jacobi–Nijenhuis manifolds

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Abstract

After a brief review on the basic notions and the principal results concerning the Jacobi manifolds, the relationship between homogeneous Poisson manifolds and conformal Jacobi manifolds, and also the compatible Jacobi manifolds, we give a generalization of some of these results needed for the contents of this paper. We introduce the notion of Jacobi–Nijenhuis structure and we study the relation between Jacobi–Nijenhuis manifolds and homogeneous Poisson–Nijenhuis manifolds. We present a local classification of homogeneous Poisson–Nijenhuis manifolds and we establish some local models of Jacobi–Nijenhuis manifolds.

MSC: 53D05; 53D10; 53D17

Keywords: Jacobi manifold; Homogeneous Poisson manifold; Jacobi–Nijenhuis manifold; Homogeneous Poisson–Nijenhuis manifold

1. Introduction

The notion of *Jacobi–Nijenhuis structure* was introduced in [17] by Marrero et al. and includes, as a particular case, that of *weak Poisson–Nijenhuis structure* presented in [18]. In this paper we propose a stricter definition of this notion, which generalizes in a natural manner that of *Poisson–Nijenhuis structure* introduced by Magri and Morosi [6,14], in order to study the completely integrable hamiltonian systems. The aim of this paper is to evidence some aspects of the local geometry of this new structure, hoping that it will play a part as important as Poisson, Jacobi and Poisson–Nijenhuis structures in the study of integrable systems.

The paper is divided into three parts.

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Paragraphs 1–3 of Section 2 (Sections 2.1–2.5) are devoted to the review and some complements of the essential definitions and results on Jacobi manifolds, conformal Jacobi manifolds, homogeneous Poisson manifolds and compatible Jacobi manifolds. In paragraph 4 we introduce the notion of *Nijenhuis operator*, while in paragraph 5 we define the notions of *Jacobi–Nijenhuis structure*, *conformal Jacobi–Nijenhuis structure* and *homogeneous Poisson–Nijenhuis structure*, and we establish a particular relation between Jacobi–Nijenhuis manifolds and homogeneous Poisson–Nijenhuis manifolds. Precisely, we prove that an one-codimensional submanifold of a homogeneous Poisson–Nijenhuis manifold, which is transverse to the homothety vector field, possesses an induced Jacobi–Nijenhuis structure (cf. Proposition 2.12), and that any Jacobi–Nijenhuis manifold can be obtained in this way (cf. Proposition 2.16).

In Section 3 (Sections 3.1–3.4), using the results of [21,23] concerning the local models of Poisson–Nijenhuis structures, we present a local classification of homogeneous Poisson–Nijenhuis manifolds.

Finally, Section 4 (Sections 4.1 and 4.2) describes some local models of Jacobi–Nijenhuis manifolds. On the neighbourhood of a generic point of a differentiable Jacobi–Nijenhuis manifold, we establish the existence of a local coordinates system in which the coefficients of the tensor fields that define the Jacobi–Nijenhuis structure are polynomials of degree less or equal to 3.

Notation: In this paper, we denote by M a C^∞-differentiable manifold of finite dimension, TM and T^*M, respectively, the tangent and cotangent bundle over M, $C^\infty(M, \boldsymbol{R})$ the space of real C^∞-differentiable functions on M, $\Omega^k(M)$, $k \in \boldsymbol{N}$, the space of exterior differentiable k-forms on M, and $\mathcal{V}^k(M)$, $k \in \boldsymbol{N}$, the space of skew-symmetric contravariant k-tensor fields on M.

For the Schouten bracket (cf. [10,25]) and the interior product of a form with a multivector field, we use the convention of sign indicated by Koszul (cf. [8,16]).

2. Part I

2.1. Jacobi manifolds

Let M be a C^∞-differentiable manifold of finite dimension. We consider on M a bivector field Λ and a vector field E which define on $C^\infty(M, \boldsymbol{R})$ the internal composition law:

$$\{f, g\} = \Lambda(df, dg) + \langle fdg - gdf, E\rangle, \qquad f, g \in C^\infty(M, \boldsymbol{R}). \tag{1}$$

It is bilinear, skew-symmetric and it verifies, for all $f, g, h \in C^\infty(M, \boldsymbol{R})$, the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

if and only if

$$[\Lambda, \Lambda] = -2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda] = 0, \tag{2}$$

where [,] denotes the Schouten bracket. When conditions (2) are verified, we say that the pair (Λ, E) defines a *Jacobi structure* on M and that (M, Λ, E) is a *Jacobi manifold*.

The bracket (1) is called the *Jacobi bracket* and the space $(C^\infty(M, \boldsymbol{R}), \{\,,\,\})$ is a local Lie algebra in the sense of Kirillov (cf. [3,5]).

In the particular case where E identically vanishes on M, conditions (2) reduce to

$$[\Lambda, \Lambda] = 0,$$

i.e. in this case, Λ endows M with a *Poisson structure*.

We denote by $\Lambda^\# : T^*M \to TM$ and $(\Lambda, E)^\# : T^*M \times \boldsymbol{R} \to TM \times \boldsymbol{R}$ the vector bundle maps associated, respectively, with Λ and (Λ, E), i.e. for all sections α, β of T^*M and for all $f \in C^\infty(M, \boldsymbol{R})$,

$$\langle \beta, \Lambda^\#(\alpha)\rangle = \Lambda(\alpha, \beta) \tag{3}$$

and

$$(\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E\rangle). \tag{4}$$

These maps can be seen, respectively, as homomorphisms of $C^\infty(M, \boldsymbol{R})$-modules; $\Lambda^\# : \Omega^1(M) \to \mathcal{V}^1(M)$ and $(\Lambda, E)^\# : \Omega^1(M) \times C^\infty(M, \boldsymbol{R}) \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$.

Finally, with any function $f \in C^\infty(M, \boldsymbol{R})$, we associate the vector field

$$X_f = \Lambda^\#(df) + fE \tag{5}$$

which is called the *hamiltonian vector field associated with* f.

The image of $\Lambda^\#$ and the vector field E define a completely integrable distribution on M, called the *characteristic distribution of* (M, Λ, E), (cf. [1,3,5]). This distribution defines a Stefan foliation of M whose leaves, which are generated by the hamiltonian vector fields (5), are called the *characteristic leaves of the Jacobi structure* (Λ, E) of M.

If, at every point of M, the dimension of the characteristic leaf of (Λ, E) through that point is equal to the dimension of M, the Jacobi manifold (M, Λ, E) is said to be *transitive*. According to the parity of the dimension of M, there are two kinds of transitive Jacobi manifolds:

1. If M has odd dimension, (Λ, E) is defined by a contact one-form (cf. [2,11]).
2. If M has even dimension, (Λ, E) is defined by a locally conformal symplectic structure (cf. [2,11]).

The characteristic leaves of (Λ, E) are themselves transitive Jacobi manifolds (cf. [2,11]).

Given a Jacobi structure (Λ, E) on M, the space $\Omega^1(M) \times C^\infty(M, \boldsymbol{R})$ is endowed with a Lie algebra structure whose bracket

$$\{\,,\,\} : (\Omega^1(M) \times C^\infty(M, \boldsymbol{R}))^2 \to \Omega^1(M) \times C^\infty(M, \boldsymbol{R}) \tag{6}$$

is defined, for all $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \boldsymbol{R})$, by

$$\{(\alpha, f), (\beta, g)\} := (\gamma, h), \tag{7}$$

where

$$\gamma := L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha, \beta)) + fL_E\beta - gL_E\alpha - i_E(\alpha \wedge \beta), \tag{8}$$

$$h := -\Lambda(\alpha, \beta) + \Lambda(\alpha, dg) - \Lambda(\beta, df) + \langle fdg - gdf, E\rangle, \tag{9}$$

(L denotes the Lie derivative operator) (cf. [4]). When E identically vanishes on M, i.e. Λ is a Poisson tensor on M, the projection of (6) on $\Omega^1(M)$ coincides with the bracket associated with Λ that endows this space with a Lie algebra structure (cf. [6,27]).

Let $a \in C^\infty(M, \boldsymbol{R})$ be a function that never vanishes on M, and $\{\,,\,\}^a : C^\infty(M, \boldsymbol{R}) \times C^\infty(M, \boldsymbol{R}) \to C^\infty(M, \boldsymbol{R})$ a new internal composition law on $C^\infty(M, \boldsymbol{R})$, bilinear and skew-symmetric, given, for each pair $(f, g) \in C^\infty(M, \boldsymbol{R}) \times C^\infty(M, \boldsymbol{R})$, by

$$\{f, g\}^a := \frac{1}{a}\{af, ag\}. \tag{10}$$

This law endows the space $C^\infty(M, \boldsymbol{R})$ with a new Jacobi bracket that defines a new Jacobi structure (Λ^a, E^a) on M, which is said to be *a-conformal* to the initially given one. The structures (Λ, E) and (Λ^a, E^a) are said to be *conformally equivalent*. One has

$$\Lambda^a = a\Lambda \quad \text{and} \quad E^a = \Lambda^\#(da) + aE. \tag{11}$$

The equivalence class of the Jacobi structures on M that are conformally equivalent to a given Jacobi structure is called the *conformal Jacobi structure of M*.

Let (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) be two Jacobi manifolds and $\phi : M_1 \to M_2$ a differentiable map. If Λ_1 and E_1 are projectable by ϕ on M_2 and their projections are, respectively, Λ_2 and E_2, i.e. $\phi_*\Lambda_1 = \Lambda_2$ and $\phi_*E_1 = E_2$, then $\phi : M_1 \to M_2$ is said to be a *Jacobi morphism* or a *Jacobi map*. When $\phi : M_1 \to M_2$ is a diffeomorphism, the Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are said to be *equivalent*.

A map $\phi : M_1 \to M_2$ is called an *a-conformal Jacobi map* if there exists $a \in C^\infty(M_1, \boldsymbol{R})$ that never vanishes on M_1 such that $\phi : (M, \Lambda_1^a, E_1^a) \to (M, \Lambda_2, E_2)$ is a Jacobi map.

For a more detailed exposition of the essential properties of Jacobi manifolds, see [11,15].

2.2. Homogeneous Poisson manifolds and conformal Jacobi manifolds

In this paragraph, we present and we complete some results, needed in the sequel, due to Lichnerowicz ([11,12]), and to Dazord et al. ([2]), concerning the homogeneous Poisson manifolds and the conformal Jacobi manifolds.

Definition 2.1. A homogeneous Poisson manifold (M, Λ, T) is a Poisson manifold (M, Λ) with a vector field T on M, called the homothety vector field, such that

$$L_T\Lambda = [T, \Lambda] = -\Lambda.$$

Proposition 2.1 ([2]). *Let (M, Λ, T) be a homogeneous Poisson manifold and Σ a submanifold of M, of codimension* 1, *transverse to the homothety vector field T. Then, Σ has an induced Jacobi structure $(\Lambda_\Sigma, E_\Sigma)$ characterized by one of the following properties*:

1. *For any pair (f, g) of homogeneous functions of degree* 1 *with respect to T, defined on an open subset $\mathcal{O}$ of M, the Jacobi bracket of f and g, restricted to $\Sigma \cap \mathcal{O}$, is the restriction to $\Sigma \cap \mathcal{O}$ of the Poisson bracket of f and g.*
2. *Let $\pi : U \to \Sigma$ be the projection on Σ of a tubular neighbourhood U of Σ in M such that, for any $x \in \Sigma$, $\pi^{-1}(x)$ is a connected arc of the integral curve of T through x.*

Let a be a function on U, equal to 1 on Σ and homogeneous of degree 1 with respect to T. Then, the projection π is an a-conformal Jacobi map.

Of course, the characteristic leaves of the Jacobi structure $(\Lambda_\Sigma, E_\Sigma)$ on Σ are (at least locally) the projections on Σ, parallel to the integral curves of T, of the symplectic leaves of (M, Λ). Since these last ones are all of even dimension, one has:

1. A leaf of $(\Sigma, \Lambda_\Sigma, E_\Sigma)$ has even dimension if and only if T is not tangent to the corresponding leaf of (M, Λ). Then, the restriction of $\pi : U \to \Sigma$ to this symplectic leaf of (M, Λ) is a local diffeomorphism of this leaf of (M, Λ) onto the corresponding leaf of $(\Sigma, \Lambda_\Sigma, E_\Sigma)$.
2. A leaf of $(\Sigma, \Lambda_\Sigma, E_\Sigma)$ has odd dimension if and only if T is tangent to the corresponding leaf of (M, Λ). Then, the dimension of this leaf of $(\Sigma, \Lambda_\Sigma, E_\Sigma)$ is lower one unity than the dimension of the corresponding leaf of (M, Λ).

In order to determine, in practice, the pair $(\Lambda_\Sigma, E_\Sigma)$ we do as follows: (i) we compute the function a, equal to 1 on Σ and homogeneous of degree 1 with respect to T, i.e. $L_T a = a$; (ii) we compute the tensor fields Λ^a and E^a that define, on a tubular neighbourhood U of Σ in M, the a-conformal Jacobi structure to its Poisson structure; (iii) we denote by $\pi : U \to \Sigma$ the projection of U on Σ, parallel to the integral curves of T, and we project Λ^a and E^a on Σ by π. Since π is a Jacobi map of (U, Λ^a, E^a) onto $(\Sigma, \Lambda_\Sigma, E_\Sigma)$, we have

$$\Lambda_\Sigma = \pi_* \Lambda^a \quad \text{and} \quad E_\Sigma = \pi_* E^a. \tag{12}$$

Notice that when a Poisson manifold (M, Λ) possesses a homothety vector field T, i.e. $L_T \Lambda = -\Lambda$, this one is not unique. Each vector field of type $T + X$, where X is an infinitesimal Poisson automorphism of Λ, i.e. $L_X \Lambda = 0$, is also a homothety vector field of Λ. Let Σ be an one-codimensional submanifold of M, transverse to two different homothety vector fields of Λ. The influence of the choice of a homothety vector field of (M, Λ) on the Jacobi structure induced on Σ by the homogeneous Poisson structure of M will be studied next.

Lemma 2.1. *Let (M, Λ, T) be a homogeneous Poisson manifold, Σ an one-codimensional submanifold of M transverse to the homothety vector field T and $(\Lambda_\Sigma, E_\Sigma)$ the Jacobi structure on Σ induced by the homogeneous Poisson structure (Λ, T) of M. Then, a vector field T' on M is a homothety vector field of Λ if and only if*

$$T' = X + hT,$$

where X is a vector field tangent to Σ and h is a differentiable function such that:

$$[X, \Lambda_\Sigma] + [X, T] \wedge E_\Sigma - h\Lambda_\Sigma = -\Lambda_\Sigma, \tag{13}$$

$$[X, E_\Sigma] + [h, \Lambda_\Sigma] - (h + \langle dh, T\rangle)E_\Sigma = -E_\Sigma. \tag{14}$$

Proof. Let p be a point of Σ such that $T(p) \neq 0$ and Σ is transverse to T at p. We may suppose, restricting Σ if needed, that there exists an open neighbourhood U of p in M

which can be identified with the product $\Sigma \times I$ of the submanifold Σ and an open interval I of $\boldsymbol{R}$ containing 0. Therefore, Σ is identified with $\Sigma \times \{0\}$ and T, restricted to U, with the vector field whose projections on Σ and I are, respectively, the zero vector field and the constant vector field equal to 1, i.e. if t is the canonical coordinate on I, $T = \partial/\partial t$. Then, from Eqs. (11) and (12), it follows that

$$\Lambda|_U = \frac{1}{a}(\Lambda_\Sigma + T \wedge E_\Sigma), \tag{15}$$

where a is the homogeneous function of degree 1 with respect to T, defined on $U = \Sigma \times I$, whose restriction to Σ is equal to 1, i.e. $a(x,t) = e^t$. Also, any vector field T' on U can be written as

$$T' = X + hT,$$

where X is a vector field tangent to Σ and h is a differentiable function on U. It is easy to check that T' is a homothety vector field of Λ if and only if X and h satisfy Eqs. (13) and (14). □

Remark 2.1. Obviously, T' is transverse to Σ at p if and only if $h(p) \neq 0$. In this case, we may suppose, restricting U if needed, that h never vanishes on U.

Lemma 2.2. *Under the same hypothesis and notations as above, let $T' = X + hT$ be a homothety vector field of Λ, with h never vanishing on U. The homogeneous functions of degree 1 with respect to T', defined on U and constant on Σ, are the functions of type*

$$f(x,t) = F(x) \exp\left(\int \frac{\mathrm{d}t}{h}\right) \tag{16}$$

satisfying $L_X f = 0$, where F is an arbitrary differentiable function on Σ.

Proof. Let f be a differentiable function defined on $U = \Sigma \times I$ having the properties described above. Then, $L_{T'} f = f$ and $L_X f = 0$. We have

$$\langle df, T' \rangle = \langle df, X + hT \rangle = \langle df, X \rangle + h\langle df, T \rangle = h\frac{\partial f}{\partial t} = f.$$

Hence,

$$f(x,t) = \exp\left(\int \frac{\mathrm{d}t}{h} + \varphi(x)\right),$$

where φ is an arbitrary differentiable function independent of t. Setting $F(x) = \exp(\varphi(x))$, we get Eq. (16). □

Always in the context of the above lemmas, we denote by $\pi : U \to \Sigma$, $U = \Sigma \times I$, the first projection, which is the projection of U on Σ parallel to the integral curves of T. Let $T' = X + hT$ be a homothety vector field of (M, Λ) different from T, transverse to Σ at p, i.e. $h(p) \neq 0$, and $\pi' : U \to \Sigma$ the projection of U on Σ parallel to the integral curves of T'. After having considered the identification of U with $\Sigma \times I$ and of

T with $\partial/\partial t$, π' is the map that takes each point (x, t) of $U = \Sigma \times I$ to the unique point x' of Σ such that $(x', 0)$ and (x, t) belong to the same integral curve of T'. Since Σ is an one-codimensional submanifold of (M, Λ, T') transverse to T', it possesses a Jacobi structure $(\Lambda'_\Sigma, E'_\Sigma)$ induced by (Λ, T'), in the sense of Proposition 2.1, such that π' is an a'-conformal Jacobi map of $(U, \Lambda|_U)$ onto $(\Sigma, \Lambda'_\Sigma, E'_\Sigma)$, where a' is a homogeneous function of degree 1 with respect to T', i.e. $L_{T'}a' = a'$, defined on U and equal to 1 on Σ. Next proposition states a relationship between $(\Lambda_\Sigma, E_\Sigma)$ and $(\Lambda'_\Sigma, E'_\Sigma)$.

Proposition 2.2. *Under the same assumptions and notations as above, we get*

$$\Lambda'_\Sigma = \Lambda_\Sigma - \frac{1}{h_0} X_0 \wedge E_\Sigma \quad \textit{and} \quad E'_\Sigma = \frac{1}{h_0} E_\Sigma,$$

where h_0 and X_0 are, respectively, the restrictions of h and X to $\Sigma \times \{0\}$, identified with Σ.

Proof. Let f and g be two functions defined on a neighbourhood U_Σ of p in Σ. We denote by F and G two functions defined on a neighbourhood of $(p, 0)$ in $\Sigma \times I$, constant on each integral curve of T', whose restrictions to $\Sigma \times \{0\}$, identified with Σ, coincide with f and g, respectively. Since $\pi' : (U, \Lambda|_U) \to (\Sigma, \Lambda'_\Sigma, E'_\Sigma)$ is an a'-conformal Jacobi map, we have

$$\Lambda'_\Sigma(df, dg) = a'\Lambda(dF, dG) \quad \text{and} \quad E'_\Sigma = \pi'_*(\Lambda^\#(da')),$$

with the following convention: if the left member of the first equation is evaluated at $x \in U_\Sigma$, then the right member of this equation must be evaluated at a point (y, t) of $\Sigma \times I$ belonging to the integral curve of T' through $(x, 0)$. We choose $y = x$ and $t = 0$.

We compute dF and dG at $(x, 0)$. We have

$$dF(x, 0) = D_x F(x, 0) + \frac{\partial F}{\partial t}(x, 0)dt,$$

where $D_x F$ is the partial derivative of F with respect to the variables x on Σ. Since $F(x, 0) = f(x)$, $D_x F(x, 0) = df(x)$. Moreover, $\langle dF(x, 0), T'(x, 0)\rangle = 0$, because F is constant on the integral curves of T'. Last equality gives

$$\left\langle \frac{\partial F}{\partial t}(x, 0)dt, T(x, 0) \right\rangle = -\frac{1}{h(x, 0)} \langle df(x), X(x, 0)\rangle.$$

So,

$$dF(x, 0) = df(x) - \frac{1}{h(x, 0)} \langle df(x), X(x, 0)\rangle dt$$

and also

$$dG(x, 0) = dg(x) - \frac{1}{h(x, 0)} \langle dg(x), X(x, 0)\rangle dt.$$

Then, taking into account Eq. (15) and the fact that $\langle dt, T\rangle = 1$,

$$\begin{aligned}\Lambda'_{\Sigma(x)}(df(x), dg(x)) &= a'(x,0)\Lambda_{(x,0)}(dF(x,0), dG(x,0)) \\ &= \frac{a'(x,0)}{a(x,0)}(\Lambda_\Sigma + T \wedge E_\Sigma)_{(x,0)}\left(df(x) - \frac{1}{h(x,0)}\right. \\ &\quad \left.\times\langle df(x), X(x,0)\rangle dt, dg(x) - \frac{1}{h(x,0)}\langle dg(x), X(x,0)\rangle dt\right) \\ &= \Lambda_{\Sigma(x)}(df(x), dg(x)) - \frac{1}{h(x,0)}\langle df(x), X(x,0)\rangle \\ &\quad \times\langle dg(x), E_\Sigma(x)\rangle + \frac{1}{h(x,0)}\langle dg(x), X(x,0)\rangle\langle df(x), E_\Sigma(x)\rangle.\end{aligned}$$

So, we get

$$\Lambda'_\Sigma = \Lambda_\Sigma - \frac{1}{h_0}X_0 \wedge E_\Sigma,$$

where h_0 and X_0 denote, respectively, the restrictions of h and X to $\Sigma \times \{0\}$.

On the other hand,

$$E'_\Sigma(x) = T_{(x,0)}\pi'(\Lambda^\#_{(x,0)}(da'(x,0))).$$

But, a' as a homogeneous function of degree 1 with respect to T', equal to 1 on Σ, is of type (16). Furthermore, $\Lambda^\#_{\Sigma(x)}(da'(x,0)) = 0$ and $\langle da'(x,0), E_\Sigma(x)\rangle = 0$. Then,

$$\begin{aligned}\Lambda^\#_{(x,0)}(da'(x,0)) &= \frac{1}{a(x,0)}(\Lambda^\#_{\Sigma(x)}(da'(x,0)) + \langle da'(x,0), T\rangle E_\Sigma \\ &\quad - \langle da'(x,0), E_\Sigma\rangle T) = \frac{\partial a'}{\partial t}(x,0)E_\Sigma = \frac{a'(x,0)}{h(x,0)}E_\Sigma\end{aligned}$$

and we deduce

$$E'_\Sigma = \frac{1}{h_0}E_\Sigma. \qquad \square$$

Proposition 2.3 ([2]). *Let (M_1, Λ_1, T_1) and (M_2, Λ_2, T_2) be two homogeneous Poisson manifolds.*

1. *The product $M_1 \times M_2$ equipped with the Poisson tensor $\Lambda_1 + \Lambda_2$ and the homothety vector field $T_1 + T_2$ is a homogeneous Poisson manifold.*
2. *Let Σ_1 be an one-codimensional submanifold of M_1 transverse to T_1 and $(\Lambda_{1\Sigma_1}, E_{1\Sigma_1})$ the Jacobi structure induced on Σ_1 by the homogeneous Poisson structure (Λ_1, T_1) of M_1. Then, $\Sigma_1 \times M_2$ is an one-codimensional submanifold of $M_1 \times M_2$ transverse to $T_1 + T_2$; the bivector field $\Lambda_{\Sigma_1\times M_2}$ and the vector field $E_{\Sigma_1\times M_2}$ that define its Jacobi structure induced by the homogeneous Poisson structure $(\Lambda_1+\Lambda_2, T_1+T_2)$ of $M_1\times M_2$ are given, respectively, by the formulæ*

$$\Lambda_{\Sigma_1\times M_2} = \Lambda_{1\Sigma_1} + \Lambda_2 - T_2 \wedge E_{1\Sigma_1} \quad \text{and} \quad E_{\Sigma_1\times M_2} = E_{1\Sigma_1}.$$

Proposition 2.4 ([2]). *Let (M, Λ, T) be a homogeneous Poisson manifold, and Σ and Σ' two submanifolds of M of codimension 1 transverse to T. We assume that there exists an integral curve of T intersecting Σ at a point p and Σ' at a point p'. We provide Σ and Σ' with the Jacobi structures induced by the homogeneous Poisson structure of M, in the sense of Proposition 2.1. Then, there exists a conformal Jacobi diffeomorphism of a neighbourhood of p in Σ onto a neighbourhood of p' in Σ', mapping p to p'.*

Proposition 2.5 ([2]). *With any Jacobi manifold (M, Λ, E) we may associate a homogeneous Poisson manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ by setting $\tilde{M} = M \times \boldsymbol{R}$,*

$$\tilde{\Lambda} = e^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge E\right) \quad \text{and} \quad \tilde{T} = \frac{\partial}{\partial t},$$

where t is the canonical coordinate on the factor $\boldsymbol{R}$. Then,

1. *the projection $\pi : \tilde{M} \to M$ is a e^t-conformal Jacobi map;*
2. *the Jacobi structure induced on M, considered as an one-codimensional submanifold of $\tilde{M}$ transverse to $\tilde{T}$, by the homogeneous Poisson structure of $\tilde{M}$, in the sense of Proposition 2.1, is the one given initially.*

The manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ is called the Poissonization of the Jacobi manifold (M, Λ, E).

2.3. Compatible Jacobi structures

Generalizing the notion of compatibility of two Poisson tensors (cf. [13]), we are lead, in a natural way, to the definition of compatibility of two Jacobi structures defined on a differentiable manifold introduced in [19] by one of the authors. In this paragraph, we recall and we complete some results of [19] on compatible pairs of Jacobi structures, useful in the sequel.

Definition 2.2. Two Jacobi structures (Λ_0, E_0) and (Λ_1, E_1) defined on a differentiable manifold M are said to be compatible if $(\Lambda_0 + \Lambda_1, E_0 + E_1)$ is also a Jacobi structure on M; this fact can be expressed by

$$[\Lambda_0, \Lambda_1] = -E_0 \wedge \Lambda_1 - E_1 \wedge \Lambda_0 \quad \text{and} \quad [E_0, \Lambda_1] + [E_1, \Lambda_0] = 0.$$

Proposition 2.6 ([19]). *Let (Λ_0, E_0) and (Λ_1, E_1) be two compatible Jacobi structures on a differentiable manifold M. Then, for any $a \in C^\infty(M, \boldsymbol{R})$ that never vanishes on M, the Jacobi structures (Λ_0^a, E_0^a) and (Λ_1^a, E_1^a) a-conformal, respectively, to (Λ_0, E_0) and (Λ_1, E_1) are also compatible on M.*

Proposition 2.7 ([19]). *Two Jacobi structures (Λ_0, E_0) and (Λ_1, E_1) defined on a differentiable manifold M are compatible if and only if the homogeneous Poisson tensors $\tilde{\Lambda}_0 = e^{-t}(\Lambda_0 + (\partial/\partial t) \wedge E_0)$ and $\tilde{\Lambda}_1 = e^{-t}(\Lambda_1 + (\partial/\partial t) \wedge E_1)$, with respect to $\partial/\partial t$, associated, respectively, with (Λ_0, E_0) and (Λ_1, E_1), are compatible on $\tilde{M} = M \times \boldsymbol{R}$.*

Definition 2.3. A homogeneous bihamiltonian manifold $(M, \Lambda_0, \Lambda_1, T)$ is a differentiable manifold M equipped with a pair (Λ_0, Λ_1) of compatible Poisson tensors in the sense of

Magri, i.e. $\Lambda_0 + \Lambda_1$ is also a Poisson tensor on M, and with a vector field T such that

$$L_T\Lambda_0 = [T, \Lambda_0] = -\Lambda_0 \quad \text{and} \quad L_T\Lambda_1 = [T, \Lambda_1] = -\Lambda_1.$$

Proposition 2.8. *Let $(M, \Lambda_0, \Lambda_1, T)$ be a homogeneous bihamiltonian manifold. We denote by Σ and Σ' two submanifolds of M, of codimension 1, transverse to the homothety vector field T. We suppose that there exists an integral curve of T intersecting Σ at a point p and Σ' at a point p'. We provide Σ (respectively Σ') with the pair of compatible Jacobi structures $((\Lambda_{0\Sigma}, E_{0\Sigma}), (\Lambda_{1\Sigma}, E_{1\Sigma}))$ (respectively $((\Lambda_{0\Sigma'}, E_{0\Sigma'}), (\Lambda_{1\Sigma'}, E_{1\Sigma'}))$) induced by the homogeneous bihamiltonian structure of M. Then, there exists a conformal Jacobi diffeomorphism of a neighbourhood of p in Σ onto a neighbourhood of p' in Σ', with respect both to $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and $(\Lambda_{0\Sigma'}, E_{0\Sigma'})$, and $(\Lambda_{1\Sigma}, E_{1\Sigma})$ and $(\Lambda_{1\Sigma'}, E_{1\Sigma'})$, mapping p to p'.*

Proof. First, we remark that the Jacobi structures $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and $(\Lambda_{1\Sigma}, E_{1\Sigma})$ (respectively $(\Lambda_{0\Sigma'}, E_{0\Sigma'})$ and $(\Lambda_{1\Sigma'}, E_{1\Sigma'})$) are compatible; this is a direct result of Propositions 2.1, 2.5 and 2.7.

From Proposition 2.4, there exists a conformal Jacobi diffeomorphism ϕ_0 (respectively ϕ_1) of a neighbourhood U_0 (respectively U_1) of p in Σ onto a neighbourhood U'_0 (respectively U'_1) of p' in Σ' mapping: (i) p to p' and (ii) an a_0 (respectively a_1)-conformal Jacobi structure to $(\Lambda_{0\Sigma}, E_{0\Sigma})$ (respectively to $(\Lambda_{1\Sigma}, E_{1\Sigma})$) to $(\Lambda_{0\Sigma'}, E_{0\Sigma'})$ (respectively to $(\Lambda_{1\Sigma'}, E_{1\Sigma'})$). From the proof of Proposition 2.4 (cf. [2]), we deduce that the diffeomorphisms ϕ_0 and ϕ_1, and also the functions a_0 and a_1, coincide on $U_0 \cap U_1$. □

2.4. Nijenhuis operator

Let M be a differentiable manifold and $\mathcal{N} : \mathcal{V}^1(M)\times C^\infty(M, \boldsymbol{R}) \to \mathcal{V}^1(M)\times C^\infty(M, \boldsymbol{R})$ a $C^\infty(M, \boldsymbol{R})$-linear map given, for all pairs $(X, f) \in \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$, by

$$\mathcal{N}(X, f) = (NX + fY, \langle\gamma, X\rangle + gf), \tag{17}$$

where N is a tensor field on M of type (1,1), Y is a vector field on M, γ is a differentiable one-form on M and g is a differentiable function on M. $\mathcal{N} := (N, Y, \gamma, g)$ can be considered as a vector bundle map $\mathcal{N} : TM \times \boldsymbol{R} \to TM \times \boldsymbol{R}$. Since the space $\mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$ endowed with the bracket

$$[\,,\,] : (\mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}))^2 \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}),$$

defined, for all $((X, f), (Z, h)) \in (\mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}))^2$, by

$$[(X, f), (Z, h)] = ([X, Z], \langle dh, X\rangle - \langle df, Z\rangle),$$

is a real Lie algebra, we can determine, in a natural way, the *Nijenhuis torsion* $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$ as the $C^\infty(M, \boldsymbol{R})$-bilinear map

$$\mathcal{T}(\mathcal{N}) : (\mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}))^2 \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$$

given, for all $((X, f), (Z, h)) \in (\mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}))^2$, by

$$\begin{aligned}\mathcal{T}(\mathcal{N})((X, f), (Z, h)) = {} & [\mathcal{N}(X, f), \mathcal{N}(Z, h)] - \mathcal{N}[\mathcal{N}(X, f), (Z, h)] \\ & - \mathcal{N}[(X, f), \mathcal{N}(Z, h)] + \mathcal{N}^2[(X, f), (Z, h)].\end{aligned}$$

Definition 2.4. A $C^\infty(M, \boldsymbol{R})$-linear map $\mathcal{N} : \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}) \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$ is called a Nijenhuis operator on M if its Nijenhuis torsion $\mathcal{T}(\mathcal{N})$ identically vanishes on M.

The notion of *Nijenhuis operator* introduced above is a generalization of the notion of *Nijenhuis tensor*. We recall that a *Nijenhuis tensor* on a differentiable manifold M is a tensor field N on M of type (1,1) whose Nijenhuis torsion

$$\begin{aligned}T(N)(X, Z) &= [NX, NZ] - N[NX, Z] - N[X, NZ] + N^2[X, Z] \\ &= (L_{NX}N - NL_X N)Z, \quad (X, Z \in \mathcal{V}^1(M)),\end{aligned}$$

identically vanishes on M.

Using $\mathcal{N} := (N, Y, \gamma, g)$ we can construct on $\tilde{M} = M \times \boldsymbol{R}$ a tensor field $\tilde{N}$ of type (1,1) by setting

$$\tilde{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt, \tag{18}$$

where t is the canonical coordinate on the factor $\boldsymbol{R}$.

Proposition 2.9 ([20]). *The tensor field $\tilde{N}$ on $\tilde{M} = M \times \boldsymbol{R}$ is a Nijenhuis tensor if and only if*

$$T(N) = Y \otimes d\gamma, \tag{19}$$

$$L_N\gamma = gd\gamma, \tag{20}$$

$$L_Y N = -Y \otimes dg, \tag{21}$$

$$^{\mathrm{t}}N(dg) = L_Y\gamma + gdg, \tag{22}$$

where $T(N)$ is the Nijenhuis torsion of N, $L_N\gamma$ is the operator on M given, for all $X, Z \in \mathcal{V}^1(M)$, by

$$L_N\gamma(X, Z) = d\gamma(NX, Z) + d\gamma(X, NZ) - d(^{\mathrm{t}}N\gamma)(X, Z),$$

and $^{\mathrm{t}}N$ is the transpose of N.

It is easy to prove that conditions (19)–(22) assure that $\mathcal{N} := (N, Y, \gamma, g)$ is a Nijenhuis operator on M, and reciprocally. So, we conclude:

Proposition 2.10. *Let $\mathcal{N} : \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}) \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$ be a $C^\infty(M, \boldsymbol{R})$-linear map given by Eq. (17). Then, $\mathcal{N}$ is a Nijenhuis operator on M if and only if its associated tensor field $\tilde{N}$ on $\tilde{M}$, given by Eq. (18), is a Nijenhuis tensor on $\tilde{M}$.*

2.5. Jacobi–Nijenhuis manifolds

Let M be a differentiable manifold of finite dimension equipped with a Jacobi structure (Λ_0, E_0) and a $C^\infty(M, \boldsymbol{R})$-linear map $\mathcal{N}: \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R}) \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$, $\mathcal{N} := (N, Y, \gamma, g)$, given by Eq. (17). Then, we can consider on M the bivector field Λ_1 and the vector field E_1 characterized by

$$(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda_0, E_0)^\#. \tag{23}$$

If we ask under what conditions does the pair (Λ_1, E_1) define on M a new Jacobi structure compatible with (Λ_0, E_0), in the sense of Definition 2.2, we find (cf. [17]):

1. Λ_1 is skew-symmetric if and only if

$$\mathcal{N} \circ (\Lambda_0, E_0)^\# = (\Lambda_0, E_0)^\# \circ {}^{\mathrm{t}}\mathcal{N}, \tag{24}$$

where ${}^{\mathrm{t}}\mathcal{N}$ denotes the transpose of $\mathcal{N}$. This condition is equivalent to the following system of conditions:

$$NE_0 = \Lambda_0^\#(\gamma) + gE_0, \tag{25}$$

$$N\Lambda_0^\# - Y \otimes E_0 = \Lambda_0^{\#}\,{}^{\mathrm{t}}N + E_0 \otimes Y, \tag{26}$$

$$\langle \gamma, E_0 \rangle = 0. \tag{27}$$

Then,

$$\Lambda_1^\# = N\Lambda_0^\# - Y \otimes E_0 = \Lambda_0^{\#}\,{}^{\mathrm{t}}N + E_0 \otimes Y, \tag{28}$$

$$E_1 = NE_0 = \Lambda_0^\#(\gamma) + gE_0. \tag{29}$$

2. When Λ_1 is skew-symmetric, (Λ_1, E_1) defines a Jacobi structure on M if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M, \boldsymbol{R})$,

$$\begin{aligned}&\mathcal{T}(\mathcal{N})((\Lambda_0, E_0)^\#(\alpha, f), (\Lambda_0, E_0)^\#(\beta, h))\\ &\quad = \mathcal{N} \circ (\Lambda_0, E_0)^\#(\mathcal{C}((\Lambda_0, E_0), \mathcal{N})((\alpha, f), (\beta, h))).\end{aligned}$$

In the last expression, $\mathcal{C}((\Lambda_0, E_0), \mathcal{N})$ is the concomitant of (Λ_0, E_0) and $\mathcal{N}$ defined, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M, \boldsymbol{R})$, by

$$\begin{aligned}&\mathcal{C}((\Lambda_0, E_0), \mathcal{N})((\alpha, f), (\beta, h))\\ &\quad = \{(\alpha, f), (\beta, h)\}_1 - \{{}^{\mathrm{t}}\mathcal{N}(\alpha, f), (\beta, h)\}_0\\ &\qquad - \{(\alpha, f), {}^{\mathrm{t}}\mathcal{N}(\beta, h)\}_0 + {}^{\mathrm{t}}\mathcal{N}\{(\alpha, f), (\beta, h)\}_0,\end{aligned}$$

($\{,\}_i$ is the bracket (6) associated with (Λ_i, E_i), $i = 0, 1$).

3. When (Λ_1, E_1) is a Jacobi structure, it is compatible with (Λ_0, E_0) if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M, \boldsymbol{R})$,

$$(\Lambda_0, E_0)^\#(\mathcal{C}((\Lambda_0, E_0), \mathcal{N})((\alpha, f), (\beta, h))) = 0.$$

Hence, we introduce the following definition.

Definition 2.5. A Jacobi–Nijenhuis structure on a differentiable manifold M is defined by a Jacobi structure (Λ_0, E_0) and a Nijenhuis operator $\mathcal{N}$ that are compatible, i.e. (i) $\mathcal{N} \circ (\Lambda_0, E_0)^{\#} = (\Lambda_0, E_0)^{\#} \circ {}^{\mathrm{t}}\mathcal{N}$ and (ii) $(\Lambda_0, E_0)^{\#} \circ \mathcal{C}((\Lambda_0, E_0), \mathcal{N}) : (\Omega^1(M) \times C^\infty(M, \boldsymbol{R}))^2 \to \mathcal{V}^1(M) \times C^\infty(M, \boldsymbol{R})$ identically vanishes on M.

$(M, (\Lambda_0, E_0), \mathcal{N})$ is said to be a Jacobi–Nijenhuis manifold. $\mathcal{N}$ is called the recursion operator of $(M, (\Lambda_0, E_0), \mathcal{N})$.

Remark 2.2. The notion of Jacobi–Nijenhuis structure presented above is stricter than the one introduced in [17]. In Definition 2.5 we require that the Nijenhuis torsion $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$ identically vanishes on M, while in [17] it is only required $\mathcal{T}(\mathcal{N})$ to be null on the image of $(\Lambda_0, E_0)^{\#}$.

Let $(M, (\Lambda_0, E_0), \mathcal{N})$ be a Jacobi–Nijenhuis manifold, (Λ_1, E_1) the Jacobi structure associated with $(\Lambda_1, E_1)^{\#} = \mathcal{N} \circ (\Lambda_0, E_0)^{\#}$, which is compatible with (Λ_0, E_0), and $a \in C^\infty(M, \boldsymbol{R})$ a function that never vanishes on M. Let us consider the Jacobi structures (Λ_0^a, E_0^a) and (Λ_1^a, E_1^a) a-conformal to (Λ_0, E_0) and (Λ_1, E_1), respectively. From Proposition 2.6, (Λ_0^a, E_0^a) and (Λ_1^a, E_1^a) are compatible. One may ask if there exists a Nijenhuis operator $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$, compatible with (Λ_0^a, E_0^a), such that $(\Lambda_1^a, E_1^a)^{\#} = \mathcal{N}^a \circ (\Lambda_0^a, E_0^a)^{\#}$.

Proposition 2.11. *Under the same assumptions and notations as above, there exists a recursion operator* $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$ *of* $((\Lambda_0^a, E_0^a), (\Lambda_1^a, E_1^a))$, *where*

$$N^a = N - Y \otimes \frac{da}{a}, \qquad Y^a = Y,$$

$$\gamma^a = \gamma + {}^{\mathrm{t}}N \frac{da}{a} - \left(g + \frac{1}{a} L_Y a\right) \frac{da}{a}, \qquad g^a = g + \frac{1}{a} L_Y a.$$

Proof. Taking into account Eqs. (11), (25)–(27), we deduce the expressions written above of N^a, Y^a, γ^a and g^a. It is easy to verify that $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$ is a Nijenhuis operator. It is compatible with (Λ_0^a, E_0^a) because (Λ_1^a, E_1^a) is a Jacobi structure compatible with (Λ_0^a, E_0^a). □

The Jacobi–Nijenhuis structure $((\Lambda_0^a, E_0^a), \mathcal{N}^a)$ is said to be *a-conformal* to $((\Lambda_0, E_0), \mathcal{N})$.

Definition 2.6 ([6,7]). A Poisson–Nijenhuis manifold (M, Λ_0, N) is a Poisson manifold (M, Λ_0) equipped with a Nijenhuis tensor N compatible with Λ_0, i.e. (i) $N \Lambda_0^{\#} = \Lambda_0^{\#\mathrm{t}} N$, where ${}^{\mathrm{t}}N$ is the transpose of N, and (ii) $\Lambda_0^{\#} \circ C(\Lambda_0, N) : \Omega^1(M) \times \Omega^1(M) \to \mathcal{V}^1(M)$ identically vanishes on M. We denote by $C(\Lambda_0, N)$ the concomitant of Magri–Morosi of Λ_0 and N given, for all $(\alpha, \beta) \in \Omega^1(M) \times \Omega^1(M)$, by

$$C(\Lambda_0, N)(\alpha, \beta) = \{\alpha, \beta\}_1 - \{{}^{\mathrm{t}}N\alpha, \beta\}_0 - \{\alpha, {}^{\mathrm{t}}N\beta\}_0 + {}^{\mathrm{t}}N\{\alpha, \beta\}_0,$$

($\{,\}_i$ is the bracket associated with Λ_i, $\Lambda_i^{\#} = N^i \Lambda_0^{\#}$, $i = 0, 1$, that endows $\Omega^1(M)$ with a Lie algebra structure).

N is called recursion operator of (M, Λ_0, N).

Definition 2.7. A Poisson–Nijenhuis manifold (M, Λ_0, N) equipped with a vector field T such that

$$L_T\Lambda_0 = [T, \Lambda_0] = -\Lambda_0 \quad \text{and} \quad L_T N = 0 \tag{30}$$

is called a homogeneous Poisson–Nijenhuis manifold.

Remark 2.3. The homogeneous Poisson–Nijenhuis manifolds are a particular class of homogeneous bihamiltonian manifolds (cf. Definition 2.3). From Eq. (30), one has $L_T\Lambda_1 = [T, \Lambda_1] = -\Lambda_1$, where Λ_1 is the Poisson tensor associated with $\Lambda_1^{\#} = N\Lambda_0^{\#}$. Moreover, T is a homothety vector field of each member of the hierarchy $(\Lambda_k, k \in \mathbf{N})$, $\Lambda_k^{\#} = N^k\Lambda_0^{\#}$, of pairwise compatible Poisson tensors generated on M by Λ_0 and N, i.e. for all $k \in \mathbf{N}$, $L_T\Lambda_k = [T, \Lambda_k] = -\Lambda_k$.

Proposition 2.12. *Let (M, Λ_0, N, T) be a homogeneous Poisson–Nijenhuis manifold and Σ an one-codimensional submanifold of M transverse to T. Then, (Λ_0, N, T) induces a Jacobi–Nijenhuis structure $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, on Σ characterized by the following properties.*

1. *$(\Lambda_{0\Sigma}, E_{0\Sigma})$ is the Jacobi structure induced on Σ by the homogeneous Poisson structure (Λ_0, T) of M, in the sense of* Proposition 2.1.
2. *$\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$ is the Nijenhuis operator induced on Σ by the (N, T) structure of M, in the sense presented next. Let $\pi : U \to \Sigma$ be the projection on Σ of a tubular neighbourhood U of Σ in M such that, for all $x \in \Sigma$, $\pi^{-1}(x)$ is a connected arc of the integral curve of T through x, and let 'a' be a differentiable function on U, that never vanishes, equal to* 1 *on Σ and homogeneous of degree 1 with respect to T, as in* Proposition 2.1. *Then, N_Σ is the tensor field of type* (1,1) *on Σ induced by N, Y_Σ is the projection of $(NT)|_\Sigma$ on $T\Sigma$ by π, γ_Σ is the one of $({}^tN(da/a))|_\Sigma$ on $T^*\Sigma$ and g_Σ is the coefficient of the component of $(NT)|_\Sigma$ in the direction of T.*

Proof. Let a be a function on U possessing the above properties. Since a is assumed to be homogeneous of degree 1 with respect to T and never vanishing on U, one has $\langle (da/a), T\rangle = 1$ and $L_T(da/a) = 0$. Then, at each point $x \in U$, $(da/a)(x)$ generates an one-dimensional subspace of T_x^*U which is the complementary of the annihilator $\langle T(x)\rangle^\circ$ of the subspace $\langle T(x)\rangle$ of T_xU generated by $T(x)$. Furthermore, $(da/a)|_\Sigma = (da)|_\Sigma$ is a section of the annihilator of $T\Sigma$.

Let us consider the projection $\pi : U \to \Sigma$ parallel to the integral curves of T. We denote by $T_\Sigma\pi : T_\Sigma U \to T\Sigma$ the vector bundle projection of $T_\Sigma U$ onto its subbundle $T\Sigma$ and ${}^tT_\Sigma\pi : T^*\Sigma \to T^*_\Sigma U$ its transpose. So,

$$T_\Sigma\pi = Id_{T_\Sigma U} - \left(T \otimes \frac{da}{a}\right)\bigg|_\Sigma, \tag{31}$$

and ${}^tT_\Sigma\pi$ is the injection that prolongs every linear form on Σ to a linear form on U that vanishes on ker $(T_\Sigma\pi) = \langle T|_\Sigma\rangle$. Then, as we have observed (cf. Section 2.2),

$$\Lambda_{0\Sigma}^{\#} = T_\Sigma\pi \circ (a\Lambda_0^{\#})|_\Sigma \circ {}^tT_\Sigma\pi, \tag{32}$$

$$E_{0\Sigma} = T_\Sigma \pi (\Lambda_0^\#(da)|_\Sigma) \overset{(31)}{=} (\Lambda_0^\#(da))|_\Sigma. \tag{33}$$

Of course, the restriction of Λ_0 to U can be written as

$$\Lambda_0 = \frac{1}{a}(\Lambda_{0\Sigma} + T \wedge E_{0\Sigma}). \tag{34}$$

On the other hand, since $L_T N = 0$, the restriction of N to U may be written as

$$N = N_\Sigma + Y_\Sigma \otimes \frac{da}{a} + T \otimes \gamma_\Sigma + g_\Sigma T \otimes \frac{da}{a}, \tag{35}$$

where N_Σ is a tensor field on Σ of type (1,1), Y_Σ is a vector field on Σ, γ_Σ is a one-form on Σ and g_Σ is a differentiable function on Σ. Since the restriction of $T_\Sigma \pi : T_\Sigma U \to T\Sigma$ to the horizontal subbundle $T\Sigma$ of $T_\Sigma U$, denoted by $(T_\Sigma \pi)_{\mathrm{h}}$, is a bijection, $N|_\Sigma : T_\Sigma U \to T_\Sigma U$ induces on Σ a tensor field of type (1,1) defined by $T_\Sigma \pi \circ N|_\Sigma \circ (T_\Sigma \pi)_{\mathrm{h}}^{-1}$. It is not hard to verify that this one is just N_Σ, i.e.

$$N_\Sigma = T_\Sigma \pi \circ N|_\Sigma \circ (T_\Sigma \pi)_{\mathrm{h}}^{-1}. \tag{36}$$

Moreover, Y_Σ can be seen as the projection of $(NT)|_\Sigma$ on $T\Sigma$, i.e.

$$Y_\Sigma = T_\Sigma \pi((NT)|_\Sigma) \overset{(31)}{=} (NT)|_\Sigma - (i((NT)|_\Sigma)(da)|_\Sigma) T|_\Sigma, \tag{37}$$

γ_Σ as the projection of $({}^{\mathrm{t}}N(da/a))|_\Sigma$ on $T^*\Sigma$, i.e.

$$\gamma_\Sigma = \left({}^{\mathrm{t}}N\frac{da}{a}\right)\bigg|_\Sigma - \left\langle \left({}^{\mathrm{t}}N\frac{da}{a}\right)\bigg|_\Sigma, T|_\Sigma \right\rangle da|_\Sigma, \tag{38}$$

and g_Σ as the coefficient of the component of $(NT)|_\Sigma$ in the direction of $T|_\Sigma$, i.e.

$$g_\Sigma = \left\langle \left(\frac{da}{a}\right)\bigg|_\Sigma, (NT)|_\Sigma \right\rangle. \tag{39}$$

Hence, from $N = N_\Sigma + Y_\Sigma \otimes (da/a) + T \otimes \gamma_\Sigma + g_\Sigma T \otimes (da/a)$ we define on Σ a $C^\infty(\Sigma, \boldsymbol{R})$-linear operator $\mathcal{N}_\Sigma : \mathcal{V}^1(\Sigma) \times C^\infty(\Sigma, \boldsymbol{R}) \to \mathcal{V}^1(\Sigma) \times C^\infty(\Sigma, \boldsymbol{R})$ by setting, for all $(X, f) \in \mathcal{V}^1(\Sigma) \times C^\infty(\Sigma, \boldsymbol{R})$,

$$\mathcal{N}_\Sigma(X, f) = (N_\Sigma X + f Y_\Sigma, \langle \gamma_\Sigma, X\rangle + g_\Sigma f). \tag{40}$$

Clearly, the tensor field N on U can be consider as the tensor field associated with $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, in the sense of Section 2.4. Then, Proposition 2.10 implies that $\mathcal{N}_\Sigma$ is a Nijenhuis operator on Σ. We are going to verify its compatibility with the Jacobi structure $(\Lambda_{0\Sigma}, E_{0\Sigma})$ of Σ. From Definition 2.5, the required conditions are: (i) $\mathcal{N}_\Sigma \circ (\Lambda_{0\Sigma}, E_{0\Sigma})^\# = (\Lambda_{0\Sigma}, E_{0\Sigma})^\# \circ {}^{\mathrm{t}}\mathcal{N}_\Sigma$ and (ii) the map $(\Lambda_{0\Sigma}, E_{0\Sigma})^\# \circ \mathcal{C}((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$ identically vanishes on Σ. But, after a long computation, we may confirm that the above mentioned conditions hold if and only if the tensor fields Λ_0 and N (cf., respectively, formulæ (34) and (35)) verify

$$N\Lambda_0^\# = \Lambda_0^{\#}{}^{\mathrm{t}}N \quad \text{and} \quad \Lambda_0^\# \circ C(\Lambda_0, N) = 0. \tag{41}$$

Since (Λ_0, N) is a Poisson–Nijenhuis structure on M, from Definition 2.6, Eq. (41) holds. Consequently, conditions (i) and (ii) also hold, and $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and $\mathcal{N}_\Sigma$ are compatible on Σ. □

Remark 2.4. Let $((\Lambda_k, k \in \boldsymbol{N}), T)$, $\Lambda_k^\# = N^k \Lambda_0^\#$, be the hierarchy of pairwise compatible Poisson tensors, homogeneous with respect to T, generated on M by (Λ_0, N) (cf. Remark 2.3). Each member (Λ_k, T) of this hierarchy induces on Σ a Jacobi structure $(\Lambda_{k\Sigma}, E_{k\Sigma})$, in the sense of Proposition 2.1. Hence, we obtain on Σ a sequence $((\Lambda_{k\Sigma}, E_{k\Sigma}), k \in \boldsymbol{N})$ of Jacobi structures. It is easy to verify that they are pairwise compatible and that, for all $k \in \boldsymbol{N}$, $(\Lambda_{k\Sigma}, E_{k\Sigma})$ coincides with the structure defined by

$$(\Lambda_{k\Sigma}, E_{k\Sigma})^\# = \mathcal{N}_\Sigma^k \circ (\Lambda_{0\Sigma}, E_{0\Sigma})^\#.$$

As in Section 2.2, we remark that when a Poisson–Nijenhuis manifold (M, Λ_0, N) possesses a vector field T verifying Eq. (30), this one is not unique; all the vector fields of type $T + X$, where X is an infinitesimal Poisson automorphism of Λ_0 such that $L_X N = 0$, also verify Eq. (30). Let Σ be an one-codimensional submanifold of M transverse to two different homothety vector fields T and T' of Λ_0 such that $L_T N = 0$ and $L_{T'} N = 0$. In Section 2.2, we studied the influence of the choice of a such vector field on the Jacobi structure induced on Σ by the homogeneous Poisson structure of M. Next, we are going to study the influence of this choice on the Nijenhuis operator induced on Σ by the Nijenhuis tensor of M.

Lemma 2.3. *Let (M, Λ_0, N, T) be a homogeneous Poisson–Nijenhuis manifold, Σ a submanifold of M of codimension 1, transverse to T, and $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma), \mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, the Jacobi–Nijenhuis structure induced on Σ by the homogeneous Poisson–Nijenhuis structure (Λ_0, N, T) of M, in the sense of Proposition 2.12. Then, a vector field T' on M verifies Eq. (30) if and only if it is of the type*

$$T' = X + hT,$$

where X is a vector field tangent to Σ and h is a differentiable function verifying Eqs. (13) and (14) and, also, the following:

$$L_X N_\Sigma + Y_\Sigma \otimes Dh + [X, T] \otimes \gamma_\Sigma + ([X, Y_\Sigma] + \langle dh, T\rangle Y_\Sigma + g_\Sigma [X, T]) \otimes \frac{da}{a} = 0, \tag{42}$$

$$-{}^{t}N_\Sigma Dh + i(X) d\gamma_\Sigma + D(\langle \gamma_\Sigma, X\rangle) - \langle dh, T\rangle \gamma_\Sigma + g_\Sigma Dh = 0, \tag{43}$$

$$-L_{Y_\Sigma} h + L_X g_\Sigma + \langle d(\langle \gamma_\Sigma, X\rangle), T\rangle = 0, \tag{44}$$

where D denotes the partial derivative with respect to the variables on Σ.

Proof. We recall the proof of Lemma 2.1 and we require $T' = X + hT$ also satisfies $L_{T'} N = 0$. Taking into account Eq. (35), we verify that $L_{T'} N = 0$ if and only if X and h fulfill Eqs. (42)–(44). □

Proposition 2.13. *Let (M, Λ_0, N, T) be a homogeneous Poisson–Nijenhuis manifold, Σ an one-codimensional submanifold of M transverse to T, and $T' = X + hT$ another vector field on M transverse to Σ such that (Λ_0, N, T') also defines a homogeneous*

Poisson–Nijenhuis structure on M. Let us endow Σ with the Jacobi–Nijenhuis structures $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, and $((\Lambda'_{0\Sigma}, E'_{0\Sigma}), \mathcal{N}'_\Sigma)$, $\mathcal{N}'_\Sigma := (N'_\Sigma, Y'_\Sigma, \gamma'_\Sigma, g'_\Sigma)$, induced, respectively, by the homogeneous Poisson–Nijenhuis structures (Λ_0, N, T) and (Λ_0, N, T') of M, in the sense of Proposition 2.12. Then,

$$\Lambda'_{0\Sigma} = \Lambda_{0\Sigma} - \frac{1}{h_0} X_0 \wedge E_{0\Sigma} \quad \textit{and} \quad E'_{0\Sigma} = \frac{1}{h_0} E_{0\Sigma}, \tag{45}$$

$$N'_\Sigma = N_\Sigma - \frac{1}{h_0} X_0 \otimes \gamma_\Sigma, \tag{46}$$

$$Y'_\Sigma = N_\Sigma X_0 - \frac{1}{h_0} \langle \gamma_\Sigma, X_0 \rangle X_0 + h_0 Y_\Sigma - g_\Sigma X_0, \tag{47}$$

$$\gamma'_\Sigma = \frac{1}{h_0} \gamma_\Sigma, \tag{48}$$

$$g'_\Sigma = g_\Sigma + \frac{1}{h_0} \langle \gamma_\Sigma, X_0 \rangle, \tag{49}$$

where X_0 and h_0 are, respectively, the restrictions of X and h on Σ.

Proof. The formulæ (45) are the result of Proposition 2.2. In order to prove Eqs. (46)–(49), we consider the same identifications as in the proofs of Lemmas 2.1 and 2.3 and Propositions 2.2 and 2.12. Let $\pi' : U \to \Sigma$ be the projection parallel to the integral curves of T' and a' a homogeneous function of degree 1 with respect to T', defined on U, and equal to 1 on Σ (cf. Lemma 2.2). We denote by $T_\Sigma \pi' : T_\Sigma U \to T\Sigma$ the vector bundle projection of $T_\Sigma U$ onto its horizontal subbundle $T\Sigma$ associated with π'. We remark that

$$\left.\frac{da'}{a'}\right|_\Sigma = \frac{1}{h_0} \left.\frac{da}{a}\right|_\Sigma,$$

where a is the homogeneous function of degree 1 with respect to T, considered in the above mentioned proofs, and also that

$$\begin{aligned} T_\Sigma \pi' &= Id_{T_\Sigma U} - \left.\left(T' \otimes \frac{da'}{a'}\right)\right|_\Sigma \\ &= Id_{T_\Sigma U} - (X_0 + h_0 T|_\Sigma) \otimes \frac{1}{h_0} \left.\frac{da}{a}\right|_\Sigma \overset{(31)}{=} T_\Sigma \pi - \frac{1}{h_0} X_0 \otimes \left.\frac{da}{a}\right|_\Sigma . \end{aligned}$$

From the geometric interpretation of the tensor fields that define on Σ the Nijenhuis operator induced by the Nijenhuis tensor of M (cf. Proposition 2.12), and considering also the identifications already made, one has

$$N'_\Sigma = T_\Sigma \pi' \circ N|_\Sigma \circ (T_\Sigma \pi')_{\rm h}^{-1}, \qquad Y'_\Sigma = T_\Sigma \pi'((NT')|_\Sigma),$$

$$\gamma'_\Sigma = \left.\left({}^{\rm t}N \frac{da'}{a'}\right)\right|_\Sigma - \left\langle \left.\left({}^{\rm t}N \frac{da'}{a'}\right)\right|_\Sigma, T|_\Sigma \right\rangle \left.\frac{da}{a}\right|_\Sigma, \qquad g'_\Sigma = \left\langle \left.\frac{da'}{a'}\right|_\Sigma, (NT')|_\Sigma \right\rangle .$$

Taking into account Eq. (35), the computation of the above formulæ yields Eqs. (46)–(49). □

Proposition 2.14. *Let* (M, Λ_0, N, T) *and* (M', Λ'_0, N', T') *be two homogeneous Poisson Nijenhuis manifolds.*

1. *The product* $M \times M'$ *endowed with* $(\Lambda_0 + \Lambda'_0, N + N', T + T')$ *is a homogeneous Poisson–Nijenhuis manifold.*
2. *Let* Σ *be an one-codimensional submanifold of* M *transverse to* T *and* $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, *the Jacobi–Nijenhuis structure induced on* Σ *by the homogeneous Poisson–Nijenhuis structure* (Λ_0, N, T) *of* M, *in the sense of* Proposition 2.12. *Then*: (i) $\Sigma \times M'$ *is an one-codimensional submanifold of* $M \times M'$ *transverse to* $T + T'$; (ii) *if* $((\Lambda_{0\Sigma\times M'}, E_{0\Sigma\times M'}), \mathcal{N}_{\Sigma\times M'})$, $\mathcal{N}_{\Sigma\times M'} := (N_{\Sigma\times M'}, Y_{\Sigma\times M'}, \gamma_{\Sigma\times M'}, g_{\Sigma\times M'})$, *denotes the Jacobi–Nijenhuis structure induced on* $\Sigma \times M'$ *by the homogeneous Poisson–Nijenhuis structure* $(\Lambda_0 + \Lambda'_0, N + N', T + T')$ *of* $M \times M'$, *its tensor fields are given, respectively, by the formulæ*

$$\Lambda_{0\Sigma\times M'} = \Lambda_{0\Sigma} + \Lambda'_0 - T' \wedge E_{0\Sigma} \quad \text{and} \quad E_{0\Sigma\times M'} = E_{0\Sigma}, \tag{50}$$

$$N_{\Sigma\times M'} = N_\Sigma + N' - T' \otimes \gamma_\Sigma, \tag{51}$$

$$Y_{\Sigma\times M'} = Y_\Sigma + (N' - g_\Sigma Id_{TM'})T', \tag{52}$$

$$\gamma_{\Sigma\times M'} = \gamma_\Sigma, \tag{53}$$

$$g_{\Sigma\times M'} = g_\Sigma. \tag{54}$$

Proof. We are only going to prove Eqs. (51)–(54); the first part and the fact that $\Sigma \times M'$ is an one-codimensional submanifold of $M \times M'$ transverse to $T + T'$ are obvious; formulæ (50) are the result of Proposition 2.3.

Let U and a be, respectively, the tubular neighbourhood of Σ in M and the homogeneous function of degree 1 with respect to T defined on U and equal to 1 on Σ that we have considered in order to construct the Jacobi–Nijenhuis structure $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$ induced on Σ by the homogeneous Poisson–Nijenhuis structure (Λ_0, N, T) of M (cf. Proposition 2.12). Now, we take the submanifold $\Sigma \times M'$ of $M \times M'$ and the tubular neighbourhood $U \times M'$ of $\Sigma \times M'$ in $M \times M'$, and we extend the function a (initially defined on U) on $U \times M'$ by imposing a to be constant on each section of type $\{x\} \times M'$, $x \in U$. Of course, the extended function a is equal to 1 on $\Sigma \times M'$ and it is homogeneous of degree 1 with respect to $T + T'$.

Let $\pi : U \times M' \to \Sigma \times M'$ be the projection parallel to the integral curves of $T + T'$. We denote by $T_{\Sigma\times M'}\pi : T_{\Sigma\times M'}(U \times M') \to T(\Sigma \times M')$ the vector bundle projection of $T_{\Sigma\times M'}(U \times M') = T_\Sigma U \oplus TM'$ onto its subbundle $T(\Sigma \times M') = T\Sigma \oplus TM'$. We have

$$\begin{aligned} T_{\Sigma\times M'}\pi &= Id_{T_{\Sigma\times M'}(U\times M')} - \left((T + T') \otimes \frac{da}{a}\right)\Bigg|_{\Sigma\times M'} \\ &= Id_{T_\Sigma U} + Id_{TM'} - \left(T \otimes \frac{da}{a}\right)\Bigg|_{\Sigma\times M'} - \left(T' \otimes \frac{da}{a}\right)\Bigg|_{\Sigma\times M'}, \end{aligned} \tag{55}$$

and we remark that the restriction of $T_{\Sigma\times M'}\pi$ to the horizontal subbundle $T(\Sigma\times M')$ of $T_{\Sigma\times M'}(U\times M')$, denoted by $(T_{\Sigma\times M'}\pi)_{\rm h}$, is the identity. From Proposition 2.12,

$$N_{\Sigma\times M'} = T_{\Sigma\times M'}\pi\circ(N+N')|_{\Sigma\times M'}\circ(T_{\Sigma\times M'}\pi)_{\rm h}^{-1},$$

$$Y_{\Sigma\times M'} = T_{\Sigma\times M'}\pi(((N+N')(T+T'))|_{\Sigma\times M'}),$$

$$\gamma_{\Sigma\times M'} = \left({}^{\rm t}(N+N')\frac{da}{a}\right)\bigg|_{\Sigma\times M'} - \left\langle\left({}^{\rm t}(N+N')\frac{da}{a}\right)\bigg|_{\Sigma\times M'},(T+T')|_{\Sigma\times M'}\right\rangle da|_{\Sigma\times M'},$$

$$g_{\Sigma\times M'} = \left\langle\frac{da}{a}\bigg|_{\Sigma\times M'},((N+N')(T+T'))|_{\Sigma\times M'}\right\rangle.$$

Taking into account (55), the computation of the above formulæ yields Eqs. (51)–(54). □

Proposition 2.15. *Let (M,Λ_0,N,T) be a homogeneous Poisson–Nijenhuis manifold and let us consider two one-codimensional submanifolds Σ and Σ' of M transverse to T. We suppose that there exists an integral curve of T intersecting Σ at a point p and Σ' at a point p'. We equip Σ (respectively Σ') with the Jacobi–Nijenhuis structure $((\Lambda_{0\Sigma},E_{0\Sigma}),\mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma:=(N_\Sigma,Y_\Sigma,\gamma_\Sigma,g_\Sigma)$, (respectively $((\Lambda_{0\Sigma'},E_{0\Sigma'}),\mathcal{N}_{\Sigma'})$, $\mathcal{N}_{\Sigma'}:=(N_{\Sigma'},Y_{\Sigma'},\gamma_{\Sigma'},g_{\Sigma'})$), induced by the homogeneous Poisson–Nijenhuis structure (Λ_0,N,T) of M, in the sense of Proposition 2.12. Then, there exists a diffeomorphism of a neighbourhood of p in Σ onto a neighbourhood of p' in Σ'* that maps: (i) *a Jacobi–Nijenhuis structure, conformal to $((\Lambda_{0\Sigma},E_{0\Sigma}),\mathcal{N}_\Sigma)$*, to $((\Lambda_{0\Sigma'},E_{0\Sigma'}),\mathcal{N}_{\Sigma'})$ and (ii) p to p'.

Proof. Let $(\Lambda_{1\Sigma},E_{1\Sigma})$ (respectively $(\Lambda_{1\Sigma'},E_{1\Sigma'})$) be the Jacobi structure on Σ (respectively Σ') generated by $((\Lambda_{0\Sigma},E_{0\Sigma}),\mathcal{N}_\Sigma)$ (respectively $((\Lambda_{0\Sigma'},E_{0\Sigma'}),\mathcal{N}_{\Sigma'})$). One has that $(\Lambda_{1\Sigma},E_{1\Sigma})$ (respectively $(\Lambda_{1\Sigma'},E_{1\Sigma'})$) is compatible with $(\Lambda_{0\Sigma},E_{0\Sigma})$ (respectively $(\Lambda_{0\Sigma'},E_{0\Sigma'})$). Taking into account Remark 2.4, $(\Lambda_{1\Sigma},E_{1\Sigma})$ (respectively $(\Lambda_{1\Sigma'},E_{1\Sigma'})$) can be seen as the Jacobi structure induced on Σ (respectively Σ') by the homogeneous Poisson structure (Λ_1,T), $\Lambda_1^\# = N\Lambda_0^\#$, of M. Then, from Proposition 2.8, there exists $a\in C^\infty(\Sigma,\boldsymbol{R})$ that never vanishes on Σ, and a diffeomorphism ϕ of a neighbourhood of p in Σ onto a neighbourhood of p' in Σ' mapping : (i) the pair $((\Lambda^a_{0\Sigma},E^a_{0\Sigma}),(\Lambda^a_{1\Sigma},E^a_{1\Sigma}))$ of compatible Jacobi structures, a-conformal to $((\Lambda_{0\Sigma},E_{0\Sigma}),(\Lambda_{1\Sigma},E_{1\Sigma}))$, to $((\Lambda_{0\Sigma'},E_{0\Sigma'}),(\Lambda_{1\Sigma'},E_{1\Sigma'}))$ and (ii) p to p'.

As it was shown in Proposition 2.11, $((\Lambda^a_{0\Sigma},E^a_{0\Sigma}),(\Lambda^a_{1\Sigma},E^a_{1\Sigma}))$ possesses a recursion operator $\mathcal{N}^a_\Sigma := (N^a_\Sigma,Y^a_\Sigma,\gamma^a_\Sigma,g^a_\Sigma)$. It is not difficult to check that ϕ takes $\mathcal{N}^a_\Sigma := (N^a_\Sigma,Y^a_\Sigma,\gamma^a_\Sigma,g^a_\Sigma)$ to $\mathcal{N}_{\Sigma'} := (N_{\Sigma'},Y_{\Sigma'},\gamma_{\Sigma'},g_{\Sigma'})$, i.e. at each point x of the considered neighbourhood of p in Σ,

$$N_{\Sigma'}(\phi(x)) = T_x\phi\circ N^a_\Sigma(x)\circ(T_x\phi)^{-1},\qquad Y_{\Sigma'}(\phi(x)) = T_x\phi(Y^a_\Sigma(x)),$$

$$\gamma_{\Sigma'}(\phi(x)) = ({}^{\rm t}T_x\phi)^{-1}(\gamma^a_\Sigma(x)),\qquad g_{\Sigma'}(\phi(x)) = g^a_\Sigma(x).$$

So, ϕ maps the Jacobi–Nijenhuis structure $((\Lambda^a_{0\Sigma}, E^a_{0\Sigma}), \mathcal{N}^a_\Sigma)$ to $((\Lambda_{0\Sigma'}, E_{0\Sigma'}), \mathcal{N}_{\Sigma'})$. □

Proposition 2.16. *With any Jacobi–Nijenhuis manifold* $(M, (\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, *we may associate a homogeneous Poisson–Nijenhuis manifold* $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ *by setting*

$$\tilde{M} = M \times \boldsymbol{R}, \qquad \tilde{\Lambda}_0 = e^{-t}\left(\Lambda_0 + \frac{\partial}{\partial t} \wedge E_0\right),$$

$$\tilde{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt, \qquad \tilde{T} = \frac{\partial}{\partial t},$$

where t is the canonical coordinate on the factor $\boldsymbol{R}$.

The Jacobi–Nijenhuis structure induced on M, considered as an one-codimensional submanifold of $\tilde{M}$ transverse to $\tilde{T}$, by the homogeneous Poisson–Nijenhuis structure of $\tilde{M}$, in the sense of Proposition 2.12, is the one given initially.

Proof. The facts that $(\tilde{\Lambda}_0, \tilde{T})$ endows $\tilde{M}$ with a homogeneous Poisson structure and that $\tilde{N}$ is a Nijenhuis tensor on $\tilde{M}$ are well known, respectively, from Propositions 2.5 and 2.10. So, it is enough to check the compatibility of these structures; condition $L_{\tilde{T}}\tilde{N} = 0$ obviously holds.

It is easy to prove that

$$\tilde{N}\tilde{\Lambda}_0^{\#} = \tilde{\Lambda}_0^{\#}\,{}^{\mathrm{t}}\tilde{N}$$

if and only if relations (25)–(27) hold. Hence,

$$\mathcal{N} \circ (\Lambda_0, E_0)^{\#} = (\Lambda_0, E_0)^{\#} \circ {}^{\mathrm{t}}\mathcal{N} \Leftrightarrow \tilde{N}\tilde{\Lambda}_0^{\#} = \tilde{\Lambda}_0^{\#}\,{}^{\mathrm{t}}\tilde{N}.$$

On the other hand, when Eqs. (25)–(27) are satisfied, we can prove that

$$(\Lambda_0, E_0)^{\#} \circ \mathcal{C}((\Lambda_0, E_0), \mathcal{N}) = 0 \Leftrightarrow \tilde{\Lambda}_0^{\#} \circ C(\tilde{\Lambda}_0, \tilde{N}) = 0.$$

Therefore, from the compatibility of (Λ_0, E_0) with $\mathcal{N}$, we deduce the compatibility of $\tilde{\Lambda}_0$ with $\tilde{N}$.

The proof of the second part of this proposition presents no difficulty. □

Remark 2.5. If $(M, (\Lambda_0, E_0), \mathcal{N})$ is a Jacobi–Nijenhuis manifold in the sense of the definition given in [17], i.e. the torsion $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$ only vanishes on the image of $(\Lambda_0, E_0)^{\#}$, then $(\tilde{\Lambda}_0, \tilde{N})$ defines a weak Poisson–Nijenhuis structure on $\tilde{M}$ in the sense of [18], i.e. the Nijenhuis torsion $T(\tilde{N})$ of $\tilde{N}$ only vanishes on the image of $\tilde{\Lambda}_0^{\#}$.

From Proposition 2.16 and Remark 2.4 we conclude, as for the Poisson–Nijenhuis manifolds, the following theorem.

Theorem 2.1 ([17]). *A Jacobi–Nijenhuis structure $((\Lambda_0, E_0), \mathcal{N})$ on a differentiable manifold M generates a hierarchy $((\Lambda_k, E_k), k \in \boldsymbol{N})$ of pairwise compatible Jacobi structures* *on M. For all $k \in \boldsymbol{N}$, (Λ_k, E_k) is the Jacobi structure associated with the vector bundle map $(\Lambda_k, E_k)^{\#} : T^*M \times \boldsymbol{R} \to TM \times \boldsymbol{R}$, $(\Lambda_k, E_k)^{\#} = \mathcal{N}^k \circ (\Lambda_0, E_0)^{\#}$.*

Furthermore, for all $k, l \in \boldsymbol{N}$, the pair $((\Lambda_k, E_k), \mathcal{N}^l)$ *defines a Jacobi–Nijenhuis structure on M.*

3. Part II

In this part of our work, we will establish some local models of homogeneous Poisson–Nijenhuis structures (cf. Definition 2.7). We apply the technic developed in [26] for the local classification of pairs of compatible symplectic forms, and we lean on the results established in [21] and [23], by one of the authors, concerning the construction of canonical forms of Poisson–Nijenhuis structures.

3.1. The regular locus of N

Let M be a differentiable manifold. We denote by $\boldsymbol{K}_M[\lambda]$ the algebra of polynomials of one variable with coefficients in the ring $\mathcal{A}(M, \boldsymbol{K})$ of the C^∞-differentiable functions, if M is a real manifold, or of the holomorphic functions on M, if M is a complex manifold. A polynomial P of $\boldsymbol{K}_M[\lambda]$ is said to be *irreducible* if it is irreducible at each point of M, and two polynomials P and Q of $\boldsymbol{K}_M[\lambda]$ are said to be *relatively prime* if they are relatively prime at each point of M.

Let N be a Nijenhuis tensor on M. It defines a section of the vector bundle $\mathrm{Hom}(TM, TM) \to M$, where $\mathrm{Hom}(TM, TM)$ denotes the bundle of the endomorphisms of TM.

Definition 3.1. We say that the algebraic type of $N : M \to \mathrm{Hom}(TM, TM)$ is constant on an open neighbourhood U of a point $p \in M$, if there exist irreducible polynomials $P_1, \dots, P_r \in \boldsymbol{K}_U[\lambda]$, relatively prime, and positive integers n_{ij}, $i = 1, \dots, r$, $j = 1, \dots, s_i$, such that, at each $x \in U$, $(P_i^{n_{ij}}, i = 1, \dots, r, j = 1, \dots, s_i)$ is the family of the elementary divisors of $N(x) : T_xM \to T_xM$.

From a geometrical point of view, the algebraic type of $N : M \to \mathrm{Hom}(TM, TM)$ is constant on U if, at each $x \in U$, T_xU is expressed as a direct sum of $N(x)$—cyclic subspaces isomorphic to the $N(p)$—cyclic subspaces of T_pU.

Definition 3.2. The map $N : M \to \mathrm{Hom}(TM, TM)$ is said to be 0-deformable on U, if the family $(P_i^{n_{ij}}, i = 1, \dots, r, j = 1, \dots, s_i)$ of its elementary divisors does not depend on the point $x \in U$.

Of course, in the case where N is 0-deformable on U, its algebraic type is constant on U.

The set of points in M possessing an open neighbourhood on which the algebraic type of N is constant, is an open dense subset of M (cf. [21]).

Definition 3.3 (Conditions of regularity). A point $p \in M$ is said to be regular with respect to N if it possesses an open neighbourhood U in M such that:

1. the algebraic type of N is constant on U;
2. the subspaces

$$\mathcal{E}_x = \bigcap_{i=1}^{s} \ker df_i(x)$$

   of T_xU, $x \in U$, where $f_1, \dots, f_s$ are the functional coefficients of the irreducible factors of the characteristic polynomial $\mathcal{P}_N$ of N, define a distribution $\mathcal{E}$ of constant rank on U;
3. the algebraic type of the restriction of N to $\mathcal{E}$ is constant on U.

Definition 3.4. We call regular locus of N, and we denote by $\mathcal{R}_N$, the set of the regular points of M with respect to N.

The set $\mathcal{R}_N$ is an open dense subset of M (cf. [21]).

3.2. Decomposition of homogeneous symplectic Poisson–Nijenhuis manifolds

Let (M, Λ_0, N, T) be a homogeneous symplectic Poisson–Nijenhuis manifold, i.e. Λ_0 is nondegenerate, fact that imposes M to have even dimension, $L_T\Lambda_0 = -\Lambda_0$ and $L_TN = 0$, and let p be a point of M having an open neighbourhood U in M on which the algebraic type of N is constant. We denote by $\mathcal{P}_N$ the characteristic polynomial of N and we assume that it is written on U as a product $\mathcal{P}_N = \mathcal{P}_1 \cdot \mathcal{P}_2$ of two polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$, relatively prime, with leading coefficient 1. Let us set $N_1 = \mathcal{P}_1(N)$ and $N_2 = \mathcal{P}_2(N)$. Then, $TU = \ker N_1 \oplus \ker N_2$ and also $TU = \operatorname{Im} N_2 \oplus \operatorname{Im} N_1$, because $\ker N_1 = \operatorname{Im} N_2$ and $\ker N_2 = \operatorname{Im} N_1$. The vector bundle maps $N_i : \operatorname{Im} N_i \to \operatorname{Im} N_i$, $i = 1, 2$, are isomorphisms. Also, $T^*U = \operatorname{Im}{}^{t}N_2 \oplus \operatorname{Im}{}^{t}N_1$, where ${}^{t}N_i = \mathcal{P}_i({}^{t}N)$ is the transpose of N_i, $i = 1, 2$.

Lemma 3.1. *The vector subbundles* $\operatorname{Im} N_i$, $i = 1, 2$, *are involutive.*

Proof. Let X and Y be two sections of $\operatorname{Im} N_1$. Since $N_1 : \operatorname{Im} N_1 \to \operatorname{Im} N_1$ is an isomorphism, $X = N_1V$ and $Y = N_1W$, where V and W are also two sections of $\operatorname{Im} N_1$. Then, $[X, Y] = [N_1V, N_1W] = T(N_1)(V, W) + N_1[N_1V, W] + N_1[V, N_1W] - N_1^2[V, W]$. But, $T(N_1)(V, W) = \sum_{r=0}^{m}(\alpha_r(V)N^rW - \alpha_r(W)N^rV)$, where α_r, $r = 1, \dots, m$, are one-forms, and so $T(N_1)(V, W)$ is a section of $\operatorname{Im} N_1$, because V and W are sections of $\operatorname{Im} N_1$. Consequently, $[X, Y]$ is a section of $\operatorname{Im} N_1$, and the involutivity of $\operatorname{Im} N_1$ is proved.

Analogously, one proves the involutivity of $\operatorname{Im} N_2$. □

Then, $\operatorname{Im} N_1$ and $\operatorname{Im} N_2$ define two complementary foliations of U. Consequently, on a convenient neighbourhood of p, M is identified with a product $M' \times M''$ of two manifolds; M' (respectively M'') is represented by the set of the leaves of the foliation defined by $\operatorname{Im} N_1$ (respectively $\operatorname{Im} N_2$). Hence, $TM' = \operatorname{Im} N_2 = \ker N_1$ and $TM'' = \operatorname{Im} N_1 = \ker N_2$.

Lemma 3.2. *For all $k \in \boldsymbol{N}$, $\Lambda_k(\mathrm{Im}\,{}^t N_2, \mathrm{Im}\,{}^t N_1) = 0$, where Λ_k is the Poisson tensor associated with the vector bundle map $\Lambda_k^\# : T^*M \to TM$, $\Lambda_k^\# = N^k \Lambda_0^\#$.*

Proof. For all α, β one-forms on U,

$$\begin{aligned}\Lambda_k({}^{\mathrm{t}}N_2\alpha, {}^{\mathrm{t}}N_1\beta) &= \Lambda_k(\mathcal{P}_2({}^{\mathrm{t}}N)\alpha, \mathcal{P}_1({}^{\mathrm{t}}N)\beta) = \Lambda_k(\mathcal{P}_1({}^{\mathrm{t}}N)\mathcal{P}_2({}^{\mathrm{t}}N)\alpha, \beta)\\ &= \Lambda_k(\mathcal{P}_N({}^{\mathrm{t}}N)\alpha, \beta) = 0,\end{aligned}$$

because $\mathcal{P}_N$ is an annihilator polynomial of ${}^{\mathrm{t}}N$. □

Proposition 3.1. *Keeping the same assumptions and notations as above, the homogeneous symplectic Poisson–Nijenhuis manifold (M, Λ_0, N, T) is identified, on a neighbourhood of p, with the product $(M', \Lambda_0', N', T') \times (M'', \Lambda_0'', N'', T'')$ of homogeneous symplectic Poisson–Nijenhuis manifolds.*

Proof. From Lemma 3.2, the tensor fields Λ_k, $k \in \boldsymbol{N}$, are locally expressed as

$$\Lambda_k = \sum_{1 \le i < j \le n_1} f_{kij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{1 \le l < m \le n_2} g_{klm} \frac{\partial}{\partial y_l} \wedge \frac{\partial}{\partial y_m},$$

where $(x_1, \dots, x_{n_1})$, $n_1 = \dim M'$, (respectively $(y_1, \dots, y_{n_2})$, $n_2 = \dim M''$), is a local coordinate system of M' (respectively M''). Since Λ_k, $k \in \boldsymbol{N}$, are pairwise compatible Poisson tensors, their associated Poisson brackets $\{,\}_k$, $k \in \boldsymbol{N}$, verify the Jacobi identity and the generalized Jacobi identity. Applying these identities to the coordinate functions, we prove that, for all $k \in \boldsymbol{N}$, f_{kij}, $1 \le i < j \le n_1$, only depend on x-coordinates and g_{klm}, $1 \le l < m \le n_2$, only depend on y-coordinates (cf. [21]).

Let us set, for all $k \in \boldsymbol{N}$,

$$\Lambda_k' = \sum_{1 \le i < j \le n_1} f_{kij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \quad \text{and} \quad \Lambda_k'' = \sum_{1 \le l < m \le n_2} g_{klm} \frac{\partial}{\partial y_l} \wedge \frac{\partial}{\partial y_m}.$$

Λ_k' (respectively Λ_k''), $k \in \boldsymbol{N}$, define on M' (respectively M'') a hierarchy of pairwise compatible Poisson tensors, with Λ_0' (respectively Λ_0'') nondegenerate on M' (respectively M''), whose recursion operator N' (respectively N'') is the projection of $N|_{\mathrm{Im}\, N_2}$ (respectively $N|_{\mathrm{Im}\, N_1}$) on $\mathrm{Im}\, N_2$ (respectively $\mathrm{Im}\, N_1$). The characteristic polynomial of N' (respectively N'') is $\mathcal{P}_1$ (respectively $\mathcal{P}_2$).

From this decomposition, the homothety vector field T is written as

$$T = T' + T'',$$

where T' (respectively T'') is a vector field tangent to M' (respectively M''), i.e. in the (x, y) product coordinates of $M = M' \times M''$,

$$T' = \sum_{i=1}^{n_1} a_i(x, y) \frac{\partial}{\partial x_i} \quad \text{and} \quad T' = \sum_{l=1}^{n_2} b_l(x, y) \frac{\partial}{\partial y_l}.$$

So, $L_T\Lambda_0 = -\Lambda_0$ if and only if

$$L_{T'}\Lambda'_0 = [T', \Lambda'_0] = -\Lambda'_0, \tag{56}$$

$$L_{T''}\Lambda''_0 = [T'', \Lambda''_0] = -\Lambda''_0, \tag{57}$$

$$L_{T'}\Lambda''_0 + L_{T''}\Lambda'_0 = [T', \Lambda''_0] + [T'', \Lambda'_0] = 0, \tag{58}$$

and $L_T\Lambda_1 = -\Lambda_1$ (cf. Remark 2.3) if and only if

$$L_{T'}\Lambda'_1 = [T', \Lambda'_1] = -\Lambda'_1, \tag{59}$$

$$L_{T''}\Lambda''_1 = [T'', \Lambda''_1] = -\Lambda''_1, \tag{60}$$

$$L_{T'}\Lambda''_1 + L_{T''}\Lambda'_1 = [T', \Lambda''_1] + [T'', \Lambda'_1] = 0. \tag{61}$$

Since Λ'_0 and Λ''_0 are nondegenerate, respectively, on M' and M'', taking into account Eqs. (56), (57), (59) and (60), and the fact that $\Lambda'_1 = N'\Lambda'_0$ and $\Lambda''_1 = N''\Lambda''_0$, we conclude that

$$L_{T'}N' = 0 \quad \text{and} \quad L_{T''}N'' = 0. \tag{62}$$

So, $L_TN = 0$ if and only if

$$L_{T'}N'' + L_{T''}N' = 0. \tag{63}$$

But, the local expressions of $L_{T'}N''$ and $L_{T''}N'$ only have, respectively, terms of type $\partial/\partial x \otimes dy$ and $\partial/\partial y \otimes dx$. Then, Eq. (63) holds if and only if

$$L_{T'}N'' = 0 \quad \text{and} \quad L_{T''}N' = 0. \tag{64}$$

Consequently, conditions (58), (61) and (64) give

$$\begin{aligned} L_{T'}\Lambda''_1 + L_{T''}\Lambda'_1 &= L_{T'}N'' \cdot \Lambda''_0 + N'' \cdot L_{T'}\Lambda''_0 + L_{T''}N' \cdot \Lambda'_0 + N' \cdot L_{T''}\Lambda'_0 \\ &= N'' \cdot L_{T'}\Lambda''_0 + N' \cdot L_{T''}\Lambda'_0 = N'' \cdot L_{T'}\Lambda''_0 - N' \cdot L_{T'}\Lambda''_0 \\ &= (N'' - N') \cdot L_{T'}\Lambda''_0 = 0. \end{aligned}$$

Thus, we obtain that, out of the singular locus of N' and N'',

$$L_{T'}\Lambda''_0 = 0, \tag{65}$$

and, on account of Eq. (58),

$$L_{T''}\Lambda'_0 = 0. \tag{66}$$

After a straightforward computation, we find that, in (x, y)-coordinates, Eqs. (65) and (66) have, respectively, the matricial expressions

$$\Lambda''_0 \cdot \begin{pmatrix} \dfrac{\partial a_1}{\partial y_1} & \cdots & \dfrac{\partial a_{n_1}}{\partial y_1} \\ \vdots & & \vdots \\ \dfrac{\partial a_1}{\partial y_{n_2}} & \cdots & \dfrac{\partial a_{n_1}}{\partial y_{n_2}} \end{pmatrix} = 0 \quad \text{and} \quad \Lambda'_0 \cdot \begin{pmatrix} \dfrac{\partial b_1}{\partial x_1} & \cdots & \dfrac{\partial b_{n_2}}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial b_1}{\partial x_{n_1}} & \cdots & \dfrac{\partial b_{n_2}}{\partial x_{n_1}} \end{pmatrix} = 0.$$

Since Λ_0'' and Λ_0' are nondegenerate, respectively, on M'' and M', the above equations imply that, out of the singular locus of N' and N'', the functional coefficients a_i, $i = 1, \ldots, n_1$, of T' only depend on the x-coordinates and the functional coefficients b_l, $l = 1, \ldots, n_2$, of T'' only depend on the y-coordinates. From the continuity of a_i, $i = 1, \ldots, n_1$, and b_l, $l = 1, \ldots, n_2$, on M, the above result holds on any neighbourhood of p in M.

From Eq. (56) (respectively Eq. (57)) and Eq. (62), we deduce that T' (respectively T'') is a homothety vector field of (Λ_0', N') (respectively (Λ_0'', N'')). □

3.3. Local models of homogeneous symplectic Poisson–Nijenhuis manifolds

Let (Λ_0, N, T) be a homogeneous symplectic Poisson–Nijenhuis structure defined on a differentiable manifold M of dimension $2n$. From the results of the previous paragraph, the problem of constructing a local model of (Λ_0, N, T) reduces to the search of the normal form of these tensor fields in the particular case where $\mathcal{P}_N$ is a power of an irreducible polynomial. The possible case are:

1. $\mathcal{P}_N(\lambda) = (\lambda + f)^{2n}$;
2. $\mathcal{P}_N(\lambda) = (\lambda^2 + g\lambda + h)^n$, (this case arises if M is a real manifold).

Studying the two cases separately, we establish in [21] the following theorems.

Theorem 3.1. *Let (Λ_0, N) be a symplectic Poisson–Nijenhuis structure defined on a differentiable manifold M (real or complex) of dimension $2n$, and p a regular point of M with respect to N. If the characteristic polynomial of N is of type $\mathcal{P}_N(\lambda) = (\lambda + f)^{2n}$ and $df(p) \neq 0$, then there exists an open neighbourhood U of p in M with local coordinates $((x_j^i), y_1, y_2)$, $i = 1, \ldots, m$, $j = 1, \ldots, 2r_i$, $r_1 \geq \cdots \geq r_m$, where $y_2 = f - a$, $a = f(p)$, centered at p, in which (Λ_0, N) has the following expression:*

$$\Lambda_0 = \sum_{i=1}^{m}\left(\sum_{k=1}^{r_i} \frac{\partial}{\partial x_{2k-1}^i} \wedge \frac{\partial}{\partial x_{2k}^i}\right) + \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}, \tag{67}$$

$$N = -(y_2 + a)Id + H + \frac{\partial}{\partial y_1} \otimes \alpha - Z \otimes dy_2, \tag{68}$$

where

$$H = \sum_{i=1}^{m}\left[\sum_{k=1}^{r_i-1}\left(\frac{\partial}{\partial x_{2k-1}^i} \otimes dx_{2k+1}^i + \frac{\partial}{\partial x_{2k+2}^i} \otimes dx_{2k}^i\right)\right], \tag{69}$$

$$\alpha = dx_2^1 + \sum_{i=1}^{m}\left(\sum_{k=1}^{r_i}\left[\left(k - \frac{1}{2}\right) x_{2k}^i dx_{2k-1}^i + \left(k + \frac{1}{2}\right) x_{2k-1}^i dx_{2k}^i\right]\right), \tag{70}$$

$$Z = \frac{\partial}{\partial x_1^1} + \sum_{i=1}^{m}\left(\sum_{k=1}^{r_i}\left[\left(k + \frac{1}{2}\right) x_{2k-1}^i \frac{\partial}{\partial x_{2k-1}^i} - \left(k - \frac{1}{2}\right) x_{2k}^i \frac{\partial}{\partial x_{2k}^i}\right]\right). \tag{71}$$

If $df(p) = 0$, the above expressions do not include the y_1 and y_2 coordinates.

Idea of proof. After the determination in [21] of the canonical form of a nondegenerate bivector defined on a $2n$-dimensional vector space V and of an endomorphism of V, and also of a symplectic Poisson–Nijenhuis structure depending on a parameter whose recursion operator is nilpotent and 0-deformable, with respect to the parameter too, we construct the model of (Λ_0, N) as follows.

If $df(p) = 0$, since $p \in \mathcal{R}_N$, f is constant on U and $(\Lambda_0, N + fId)$ defines on U a symplectic Poisson–Nijenhuis structure whose recursion operator is 0-deformable and nilpotent. Then, its model is well known from the precedents and from it we easily deduce the normal form of (Λ_0, N).

If $df(p) \neq 0$, we consider the pair $(\Lambda_0, N + fId)$ of tensor fields that induces on the integral manifolds of the quotient bundle $\ker df/X_f$, where $X_f = \Lambda_0^{\#}(df)$, a symplectic Poisson–Nijenhuis structure depending parametrically on f whose recursion operator is nilpotent and 0-deformable, with respect to the parameter too. For all values of the parameter f, the model of the induced structure is well known from the previous study. From this model, we establish the normal form of (Λ_0, N), presented by Theorem 3.1. In the local expressions ((67)–(71)) of (Λ_0, N), m denotes the number of the $(N + fId)(x)$-invariant subspaces in which the quotient space $\ker df(x)/X_f(x)$, $x \in U$, is decomposed; the ith-subspace, $i = 1, \dots, m$, is decomposed into two $(N + fId)(x)$-cyclic subspaces, both of dimension r_i; $y_2 = f - a$, $a = f(p)$, and y_1 is chosen in such a way that $\partial/\partial y_1 = X_f$.

The models are completely determined by the algebraic type of N. □

When $\mathcal{P}_N(\lambda) = (\lambda + f)^{2n}$ and $df(p) \neq 0$, we find that, in the coordinates of Theorem 3.1,

$$\Lambda_1 = -(y_2 + a)\Lambda_0 + \Pi + Z \wedge \frac{\partial}{\partial y_1}, \tag{72}$$

where

$$\Pi = \sum_{i=1}^{m} \left(\sum_{k=1}^{r_i - 1} \frac{\partial}{\partial x^i_{2k-1}} \wedge \frac{\partial}{\partial x^i_{2k+2}} \right), \tag{73}$$

and that a representative of the homothety vector field T of (Λ_0, N) is the vector field

$$\boldsymbol{T} = \frac{2}{3}\frac{\partial}{\partial x^1_1} + \sum_{i=1}^{m} \left(\sum_{k=1}^{r_i} x^i_{2k-1} \frac{\partial}{\partial x^i_{2k-1}} \right) + y_1 \frac{\partial}{\partial y_1}; \tag{74}$$

it is a model of T, modulo the addition of an infinitesimal Poisson automorphism X of Λ_0 such that $L_X N = 0$. We remark that, if $df(p) = 0$, Eqs. (72) and (74) do not include coordinates y_1 and y_2.

In the case where $\mathcal{P}_N(\lambda) = (\lambda^2 + g\lambda + h)^n$, with $g^2 - 4h$ strictly negative on a neighbourhood U of p in M, the construction of the models is based on: (i) The existence on U of a complex structure J, i.e. $J^2 = -Id$ and its Nijenhuis torsion identically vanishes. J is the semi-simple part of the operator $N_0 = 2(4h - g^2)^{-1/2}N + g(4h - g^2)^{-1/2}Id$, so there exists a polynomial $Q \in \boldsymbol{K}_U[\lambda]$ with constant coefficients, because N_0 is 0-deformable, such that $J = Q(N_0)$, i.e. J is a polynomial operator of N whose coefficients are functions depending on g and h (cf. [24]); (ii) The following lemma.

Lemma 3.3 ([21]). *Under the same assumptions and notations as above, let $\bar{\Lambda}_0$ (respectively $\bar{\Lambda}_1$) be the tensor field associated with the vector bundle map $\bar{\Lambda}_0^{\#} = J\Lambda_0^{\#}$ (respectively $\bar{\Lambda}_1^{\#} = J\Lambda_1^{\#}$). Then, $\bar{\Lambda}_0$ (respectively $\bar{\Lambda}_1$) is a Poisson tensor compatible with Λ_0 (respectively Λ_1), and $\hat{\Lambda}_0 = \Lambda_0 - i\bar{\Lambda}_0$ (respectively $\hat{\Lambda}_1 = \Lambda_1 - i\bar{\Lambda}_1$) is a holomorphic complex Poisson tensor.*

Furthermore, $(\hat{\Lambda}_0, \hat{\Lambda}_1)$ is a pair of compatible holomorphic complex Poisson tensors.

We remark that the recursion operator of $(\hat{\Lambda}_0, \hat{\Lambda}_1)$ is also N that is holomorphic. Moreover, the regular locus of N, seen as a holomorphic tensor field, coincide with the one of N, seen as a real tensor field, and its characteristic polynomial is $\hat{\mathcal{P}}_N(\lambda) = (\lambda + f)^n$, where $f = (1/2)[g - \mathrm{i}(4h - g^2)^{1/2}]$ is a holomorphic function. So, there exists a neighbourhood U of p in M with local complex coordinates $((z_l^j), w_1, w_2)$, $j = 1, \dots, m$, $l = 1, \dots, 2r_j$, $r_1 \geq \cdots \geq r_m$, centered at p, in which $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ are given, respectively, by (67) and (72). If $((x_l^j), u_1, u_2; (y_l^j), v_1, v_2)$, $j = 1, \dots, m$, $l = 1, \dots, 2r_j$, $r_1 \geq \cdots \geq r_m$, is the system of real coordinates on U associated with the complex one, after making the convenient replacements in the obtained expressions of $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$, we take their real parts. Hence, we obtain a normal form of (Λ_0, Λ_1) and, consequently, of N. They are presented in next theorem.

Theorem 3.2. *Let (Λ_0, N) be a symplectic Poisson–Nijenhuis structure defined on a real differentiable manifold M of dimension 2n, and p a regular point of M with respect to N. If the characteristic polynomial of N is of type $\mathcal{P}_N(\lambda) = (\lambda^2 + g\lambda + h)^n$, with $g^2 - 4h$ locally strictly negative, then there exists an open neighbourhood U of p in M with local coordinates $((x_l^j), u_1, u_2; (y_l^j), v_1, v_2)$, $j = 1, \dots, m$, $l = 1, \dots, 2r_j$, $r_1 \geq \cdots \geq r_m$, centered at p, in which the tensors fields Λ_0 and N are expressed as follows:*

$$\Lambda_0 = \sum_{j=1}^{m}\left[\sum_{k=1}^{r_j}\frac{1}{4}\left(\frac{\partial}{\partial x_{2k-1}^j}\wedge\frac{\partial}{\partial x_{2k}^j} - \frac{\partial}{\partial y_{2k-1}^j}\wedge\frac{\partial}{\partial y_{2k}^j}\right)\right] + \frac{1}{4}\left(\frac{\partial}{\partial u_1}\wedge\frac{\partial}{\partial u_2} - \frac{\partial}{\partial v_1}\wedge\frac{\partial}{\partial v_2}\right), \tag{75}$$

$$N = -(u_2 + a)Id - (v_2 + b)J + H_x + H_y + \frac{\partial}{\partial u_1}\otimes(\alpha_x - \alpha_y) + \frac{\partial}{\partial v_1}\otimes(\alpha_x + \alpha_y) - Z_x\otimes(du_2 + dv_2) - Z_y\otimes(du_2 - dv_2), \tag{76}$$

where $a = Re\,\hat{a}$, $b = Im\,\hat{a}$, $(\hat{a} = f(p))$,

$$J = \sum_{j,l}\left(\frac{\partial}{\partial y_l^j}\otimes dx_l^j - \frac{\partial}{\partial x_l^j}\otimes dy_l^j\right) - \frac{\partial}{\partial u_1}\otimes dv_1 - \frac{\partial}{\partial u_2}\otimes dv_2 + \frac{\partial}{\partial v_1}\otimes du_1 + \frac{\partial}{\partial v_2}\otimes du_2,$$

$$H_x = \sum_{j=1}^{m}\left[\sum_{k=1}^{r_j-1}\left(\frac{\partial}{\partial x_{2k-1}^j}\otimes dx_{2k+1}^j + \frac{\partial}{\partial x_{2k+2}^j}\otimes dx_{2k}^j\right)\right],$$

$$H_y = \sum_{j=1}^{m}\left[\sum_{k=1}^{r_j-1}\left(\frac{\partial}{\partial y_{2k-1}^j}\otimes dy_{2k+1}^j + \frac{\partial}{\partial y_{2k+2}^j}\otimes dy_{2k}^j\right)\right],$$

$$\alpha_x = dx_2^1 + \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\left[\left(k-\frac{1}{2}\right)x_{2k}^j dx_{2k-1}^j + \left(k+\frac{1}{2}\right)x_{2k-1}^j dx_{2k}^j\right]\right),$$

$$\alpha_y = \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\left[\left(k-\frac{1}{2}\right)y_{2k}^j dy_{2k-1}^j + \left(k+\frac{1}{2}\right)y_{2k-1}^j dy_{2k}^j\right]\right),$$

$$Z_x = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\left[\left(k+\frac{1}{2}\right)x_{2k-1}^j\frac{\partial}{\partial x_{2k-1}^j} - \left(k-\frac{1}{2}\right)x_{2k}^j\frac{\partial}{\partial x_{2k}^j}\right]\right),$$

$$Z_y = \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\left[\left(k+\frac{1}{2}\right)y_{2k-1}^j\frac{\partial}{\partial y_{2k-1}^j} - \left(k-\frac{1}{2}\right)y_{2k}^j\frac{\partial}{\partial y_{2k}^j}\right]\right).$$

After a long computation, we show that, in the coordinates of Theorem 3.2,

$$\boldsymbol{T} = \frac{2}{3}\frac{\partial}{\partial x_1^1} + \sum_{j=1}^{m}\sum_{k=1}^{r_j}\left(x_{2k-1}^j\frac{\partial}{\partial x_{2k-1}^j} + y_{2k-1}^j\frac{\partial}{\partial y_{2k-1}^j}\right) + u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1} \tag{77}$$

is a representative of the homothety vector field T of (Λ_0, N), modulo the addition of an infinitesimal Poisson automorphism X of Λ_0 such that $L_X N = 0$.

From this study, we conclude the following theorem.

Theorem 3.3. *Let (Λ_0, N, T) be a homogeneous symplectic Poisson–Nijenhuis structure defined on a differentiable manifold M of dimension $2n$. Then, on a neighbourhood of each regular point p of M with respect to N, the model of (M, Λ_0, N, T) is a finite product of homogeneous symplectic Poisson–Nijenhuis manifolds whose recursion operator has as characteristic polynomial a power of an irreducible polynomial.*

If M is a complex manifold, the Poisson–Nijenhuis structure's model of each factor of this product is given by Theorem 3.1 and the model of the corresponding homothety vector field is given by Eq. (74), modulo the addition of an infinitesimal Poisson biautomorphism of the factor's Poisson–Nijenhuis structure.

If M is a real manifold, the Poisson–Nijenhuis structure's model of each factor of this product is given by Theorem 3.1 *or* Theorem 3.2, *according to the type of the characteristic polynomial of the factor's recursion operator, and the model of the corresponding homothety vector field is given, respectively, by* Eq. (74) *or* (77), *modulo the addition of an infinitesimal Poisson biautomorphism of the factor's Poisson–Nijenhuis structure.*

The models are completely determined by the family of the elementary divisors of N. (*We notice that each elementary divisor appears an even number of times in this family.*)

3.4. Local models of homogeneous Poisson–Nijenhuis manifolds of odd dimension

Let (Λ_0, N, T) be a homogeneous Poisson–Nijenhuis structure defined on a $(2n+1)$-dimensional differentiable manifold M, with Λ_0 of maximum rank on an open dense subset of M. Using the results on: (i) the local models of symplectic Poisson–Nijenhuis structures (see Section 3.3 and [21]); (ii) the symplectization of a Poisson–Nijenhuis structure (see [22]) and (iii) the reduction of a Poisson–Nijenhuis structure (see [28,18]), we establish in [23] the following theorem.

Theorem 3.4. *Under the same assumptions and notations as above, on a neighbourhood of each point $p \in \mathcal{R}_N$ such that* corank $\Lambda_0(p) = 1$, *the model of (M, Λ_0, N) is a product of a Poisson–Nijenhuis manifold (M', Λ'_0, N') of odd dimension $2l-1$, $l \leq n+1$, whose Nijenhuis tensor N' has a characteristic polynomial of type $\mathcal{P}_{N'}(\lambda) = (\lambda + f)^{2l-1}$, and of a symplectic Poisson–Nijenhuis manifold (M'', Λ''_0, N'').*

If p' is the projection of p on M' and $df(p') \neq 0$, then there exists an open neighbourhood U' of p' in M' with local coordinates $((x'^i_j), y')$, $i = 1, \ldots, m$, $j = 1, \ldots, 2r_i$, $r_1 \geq \cdots \geq r_m$, $y' = f - a'$, $a' = f(p')$, centered at p', such that

$$\Lambda'_0 = \sum_{i=1}^{m} \left(\sum_{k=1}^{r_i} \frac{\partial}{\partial x'^i_{2k-1}} \wedge \frac{\partial}{\partial x'^i_{2k}} \right), \tag{78}$$

$$N' = -(y' + a')Id + H' - Z' \otimes dy', \tag{79}$$

where H' and Z' are given, in these coordinates, respectively, by Eqs. (69) and (71). *If $df(p') = 0$, expressions* (78) and (79) *and also those of H' and Z' do not include coordinates $x'^m_{2r_m}$ and y'.*

If p'' is the projection of p on M'', the normal form of the tensor fields Λ''_0 and N'', on an open neighbourhood of p'' in M'', is presented by Theorem 3.3.

The model of (M, Λ_0, N) is completely determined by the family of the elementary divisors of N.

(*In formulæ* (78) and (79), *m and r_i, $i = 1, \ldots, m$, have the same meaning as in* Theorem 3.1.)

Let (x''_k), $k = 1, \ldots, \dim M''$, be a system of local coordinates of M'', centered at p'', in which (Λ''_0, N'') has the model's expression (cf. Theorem 3.3). Because of the identification $(M, \Lambda_0, N) = (M', \Lambda'_0, N') \times (M'', \Lambda''_0, N'')$ on an open neighbourhood U of p in M, the homothety vector field T is written, in the local coordinate product system $((x'^i_j), y'; x''_k)$, $i = 1, \dots, m$, $j = 1, \dots, 2r_i$, $r_1 \geq \dots \geq r_m$, $k = 1, \dots, \dim M''$, of $M = M' \times M''$, as

$$T = T' + T'',$$

where

$$T' = \sum_{i=1}^{m} \sum_{j=1}^{2r_i} a^i_j(x', y'; x'') \frac{\partial}{\partial x'^i_j} + b(x', y'; x'') \frac{\partial}{\partial y'} \quad \text{and}$$

$$T'' = \sum_{k=1}^{\dim M''} c_k(x', y'; x'') \frac{\partial}{\partial x''_k}$$

are, respectively, vector fields tangent to M' and M''. Since (Λ_0, N, T) is a homogeneous Poisson–Nijenhuis structure, $L_T \Lambda_0 = -\Lambda_0$, $L_T N = 0$ and $L_T \Lambda_1 = -\Lambda_1$, $\Lambda_1^\# = N \Lambda_0^\#$. But, $\Lambda_i = \Lambda'_i + \Lambda''_i$, $i = 0, 1$. Hence, $L_T \Lambda_0 = -\Lambda_0$ if and only if

$$L_{T'} \Lambda'_0 = [T', \Lambda'_0] = -\Lambda'_0, \tag{80}$$

$$L_{T''} \Lambda''_0 = [T'', \Lambda''_0] = -\Lambda''_0, \tag{81}$$

$$L_{T'} \Lambda''_0 + L_{T''} \Lambda'_0 = [T', \Lambda''_0] + [T'', \Lambda'_0] = 0, \tag{82}$$

and $L_T \Lambda_1 = -\Lambda_1$ if and only if

$$L_{T'} \Lambda'_1 = [T', \Lambda'_1] = -\Lambda'_1, \tag{83}$$

$$L_{T''} \Lambda''_1 = [T'', \Lambda''_1] = -\Lambda''_1, \tag{84}$$

$$L_{T'} \Lambda''_1 + L_{T''} \Lambda'_1 = [T', \Lambda''_1] + [T'', \Lambda'_1] = 0. \tag{85}$$

Since Λ''_0 is nondegenerate on M'', Eqs. (81) and (84) yield

$$L_{T''} N'' = 0. \tag{86}$$

Therefore, $L_T N = 0$ if and only if

$$L_{T'} N' + L_{T'} N'' + L_{T''} N' = 0. \tag{87}$$

Furthermore, in the coordinate product system considered above, the matricial expressions of $L_{T'} N'$, $L_{T'} N''$ and $L_{T''} N'$ are, respectively, of type:

$$L_{T'} N' = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \qquad L_{T'} N'' = \begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L_{T''} N' = \begin{pmatrix} 0 & 0 \\ \Delta & 0 \end{pmatrix}.$$

So, Eq. (87) holds if and only if

$$L_{T'} N' + L_{T'} N'' = 0 \quad \text{and} \quad L_{T''} N' = 0. \tag{88}$$

Taking into account the second condition of Eq. (88), Eq. (85) implies

$$\begin{aligned} L_{T'}\Lambda_1'' + L_{T''}\Lambda_1' &= L_{T'}N'' \cdot \Lambda_0'' + N'' \cdot L_{T'}\Lambda_0'' + L_{T''}N' \cdot \Lambda_0' + N' \cdot L_{T''}\Lambda_0' \\ &= L_{T'}N'' \cdot \Lambda_0'' + N'' \cdot L_{T'}\Lambda_0'' + N' \cdot L_{T''}\Lambda_0' = 0. \end{aligned} \tag{89}$$

Considering the local expressions of the terms of left member of Eq. (89), we conclude that this equality holds if and only if

$$L_{T'}N'' \cdot \Lambda_0'' + N' \cdot L_{T''}\Lambda_0' = 0 \quad \text{and} \quad N'' \cdot L_{T'}\Lambda_0'' = 0.$$

So, out of the singular locus of N'',

$$L_{T'}\Lambda_0'' = 0 \tag{90}$$

and, because of Eq. (82),

$$L_{T''}\Lambda_0' = 0. \tag{91}$$

After a direct computation, we find that, in the considered local coordinate product system, Eqs. (90) and (91) are expressed, in terms of matrices, respectively, as

$$\Lambda_0'' \cdot \left(\frac{\partial (a_j^i, b)}{\partial x''} \right) = 0 \tag{92}$$

and

$$\Lambda_0' \cdot \left(\frac{\partial c_k}{\partial (x', y')} \right) = 0. \tag{93}$$

Since Λ_0'' is nondegenerate on M'', Eq. (92) means that, out of the singular locus of N'', the functional coefficients of T', a_j^i, $i = 1, \ldots, m$, $j = 1, \ldots, 2r_i$, $r_1 \geq \cdots \geq r_m$, and b, only depend on the x' and y' coordinates. Because of the continuity of these functions on M, the above result hold on any neighbourhood of p in M. On the other hand, since the restriction of Λ_0' to its symplectic leaves, defined by the equation $y' =$ constant, is inversible on these leaves, Eq. (93) implies that, out of the singular locus of N'', the functional coefficients of T'', c_k, $k = 1, \ldots, \dim M''$, only depend on the y' and x'' coordinates. Because these functions are continuous on M, the above conclusion holds on any neighbourhood of p in M

Of course, T' and T'' are, respectively, homothety vector fields of (Λ_0', N') and (Λ_0'', N'').

Let S_0 be the symplectic leaf of Λ_0 through p. Since $(M, \Lambda_0, N) = (M', \Lambda_0', N') \times (M'', \Lambda_0'', N'')$, on a neighbourhood of p, and Λ_0'' is symplectic on M'', $S_0 = S_0' \times M''$, where S_0' is the symplectic leaf of Λ_0' through p'. In the product coordinates $((x_j'^i), y'; x_k'')$, $i = 1, \ldots, m$, $j = 1, \ldots, 2r_i$, $r_1 \geq \cdots \geq r_m$, $k = 1, \ldots, \dim M''$, of $M = M' \times M''$, S_0 and S_0' are determined by the same equation $y' = 0$. The functions $((x_j'^i); x_k'')$, $i = 1, \ldots, m$, $j = 1, \ldots, 2r_i$, $r_1 \geq \cdots \geq r_m$, $k = 1, \ldots, \dim M''$, define on $S_0 = S_0' \times M''$ a coordinate product system. If $T = T' + T''$ is tangent to S_0, i.e. $b(x', y'; x'') = 0$, then T' is tangent to S_0', and reciprocally. In this case, T is a homothety vector field of the symplectic Poisson–Nijenhuis structure induced on S_0 by (Λ_0, N) and so is T' for the symplectic Poisson–Nijenhuis structure induced on S'_0 by (Λ'_0, N'). The recursion operator of the latter structure is 0-deformable and its characteristic polynomial is $(\lambda + f)^{2l-2}$. Consequently, in the considered case, the local expression of T', in coordinates (x'^i_j), $i = 1, \dots, m$, $j = 1, \dots, 2r_i$, $r_1 \geq \cdots \geq r_m$, of S'_0, is

$$T' = \frac{2}{3}\frac{\partial}{\partial x'^1_1} + \sum_{i=1}^{m}\left(\sum_{k=1}^{r_i} x'^i_{2k-1}\frac{\partial}{\partial x'^i_{2k-1}}\right), \tag{94}$$

modulo the addition of an infinitesimal Poisson biautomorphism of (Λ'_0, Λ'_1), $\Lambda'^{\#}_1 = N'\Lambda'^{\#}_0$, tangent to S'_0 (cf. Eq. (74) and the remark that follows). The local expression of T'', in coordinates (x''_k), $k = 1, \dots, \dim M''$, of M'', is well determined by Theorem 3.3.

4. Part III

In the third and last part of this work, we are going to study the problem of constructing a normal form of the tensor fields of a Jacobi–Nijenhuis structure $((\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, defined on a finite dimensional differentiable manifold M. In order to establish these forms, we consider the homogeneous Poisson–Nijenhuis structure $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ defined on $\tilde{M} = M \times \boldsymbol{R}$ from $((\Lambda_0, E_0), \mathcal{N})$ (cf. Proposition 2.16). In the case where $\tilde{\Lambda}_0$ is of maximum rank on $\tilde{M}$ (or on an open dense subset of $\tilde{M}$) and $\tilde{T}$ is tangent to the symplectic leaves of $\tilde{\Lambda}_0$ (of course, this always happen when $\tilde{\Lambda}_0$ is symplectic), the local model of $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$, on an open neighbourhood of a regular point $\tilde{p}$ of $\tilde{M}$ with respect to $\tilde{N}$, is well determined, according to the parity of the dimension of $\tilde{M}$, by Theorems 3.3 and 3.4 and by formula (94). Then, taking: (i) an one-codimensional submanifold Σ of $\tilde{M}$ transverse to the homothety vector field $\tilde{T}$; (ii) a function a defined on a tubular neighbourhood $\tilde{U}$ of Σ in $\tilde{M}$, equal to 1 on Σ and homogeneous of degree 1 with respect to $\tilde{T}$, and (iii) the pair $(\tilde{\Lambda}^a_0, \tilde{E}^a_0)$ that defines on $\tilde{U}$ the Jacobi structure which is a-conformal to the Poisson structure's model, and computing: (i) the projection of $(\tilde{\Lambda}^a_0, \tilde{E}^a_0)$ on Σ parallel to the integral curves of the model of $\tilde{T}$, and (ii) from the model of $\tilde{N}$, the Nijenhuis operator induced on Σ, we obtain on Σ a Jacobi–Nijenhuis model structure (cf. Proposition 2.12), that, from Proposition 2.15, is equivalent to a Jacobi–Nijenhuis structure on M, conformal to the one given initially. In this way, we end up establishing, on a neighbourhood of a point p of M, which is the projection on M of a regular point $\tilde{p}$ of $\tilde{M}$ with respect to $\tilde{N}$, a model of a structure that is conformal to $((\Lambda_0, E_0), \mathcal{N})$, in the cases where:

1. M has odd dimension and (Λ_0, E_0) is transitive on M;
2. M has even dimension, say $2n$, and the characteristic leaf C_0 of (Λ_0, E_0) through p has odd dimension, equal to $2n - 1$, fact that imposes $\tilde{T} = \partial/\partial t$ to be tangent to the corresponding symplectic leaf of $\tilde{\Lambda}_0$ (cf. Section 2.2).

(We remark that the set of points in M that can be seen as projections of regular points of $\tilde{M}$ with respect to $\tilde{N}$, is an open dense subset of M, because $\mathcal{R}_{\tilde{N}}$ is an open dense subset of $\tilde{M}$.)

The case where M has even dimension and (Λ_0, E_0) is transitive on M is going to be treated separately in Sections 4.2.

4.1. Local models of odd-dimensional Jacobi–Nijenhuis manifolds

Let $((\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a transitive Jacobi–Nijenhuis structure defined on a $(2n+1)$-dimensional differentiable manifold M and $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ its associated homogeneous Poisson–Nijenhuis structure on $\tilde{M} = M \times \boldsymbol{R}$ (cf. Proposition 2.16). Since $\tilde{\Lambda}_0 = e^{-t}(\Lambda_0 + (\partial/\partial t) \wedge E_0)$, ($t$ is the canonical coordinate on the factor $\boldsymbol{R}$), is nondegenerate on $\tilde{M}$, on a neighbourhood of each regular point $\tilde{p} \in \tilde{M}$ with respect to $\tilde{N} = N + Y \otimes dt + (\partial/\partial t) \otimes \gamma + g\partial/\partial t \otimes dt$, the model of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ is a finite product of homogeneous symplectic Poisson–Nijenhuis manifolds whose recursion operator has as characteristic polynomial a power of an irreducible polynomial (cf. Theorem 3.3). In what follows, this decomposition of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ is going to be referred as the *"model decomposition"* of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. Let p be the projection of $\tilde{p}$ on M. Because $\tilde{T} = (\partial/\partial t)$ is transverse to M at p, at least one of the components of the decomposition of $\tilde{T}$ is transverse to M at p. Therefore, in order to construct a local model of $((\Lambda_0, E_0), \mathcal{N})$, we distinguish and we study separately the following cases:

1. The recursion operator of the homogeneous symplectic Poisson–Nijenhuis structure of the factor of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ corresponding to the considered component of $\tilde{T}$, i.e. the component that is transverse to M at p, has a characteristic polynomial of type $(\lambda + f)^{2q}$, $q \leq n+1$.
2. The recursion operator of the homogeneous symplectic Poisson–Nijenhuis structure of the factor of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ corresponding to the considered component of $\tilde{T}$, i.e. the component that is transverse to M at p, has a characteristic polynomial of type $(\lambda^2 + f\lambda + h)^q$, $q \leq n+1$, with $f^2 - 4h$ locally strictly negative. (In order to avoid any confusion, in this paragraph, we will not use g as a coefficient of the characteristic polynomial of the recursion operator because it appears as a coefficient of $\tilde{N}$.)

4.1.1. Study of Case 1

We denote by $(\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$ the factor of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ whose homothety vector field $\tilde{T}'$ is transverse to M at p, and we suppose that its recursion operator $\tilde{N}'$ has a characteristic polynomial of type $\mathcal{P}_{\tilde{N}'}(\lambda) = (\lambda + f)^{2q}$, $q \leq n+1$. Then, on a neighbourhood of $\tilde{p}$ in $\tilde{M}$, $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T}) = (\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}') \times (\tilde{M}'', \tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$, where $(\tilde{M}'', \tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ is the product of the other factors of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. If $\tilde{p}'$ and $\tilde{p}''$ are, respectively, the projections of $\tilde{p}$ on $\tilde{M}'$ and $\tilde{M}''$, the normal form of $(\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$, on a neighbourhood of $\tilde{p}'$ in $\tilde{M}'$, is given by Theorem 3.1 and Eq. (74), and the one of $(\tilde{M}'', \tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$, on a neighbourhood of $\tilde{p}''$ in $\tilde{M}''$, by Theorem 3.3.

Now, we suppose that $df(\tilde{p}') \neq 0$, and we consider a local coordinate system $((\tilde{x}'^{i}_{j}), \tilde{y}'_1, \tilde{y}'_2)$, $i = 1, \dots, m$, $j = 1, \dots, 2r_i$, $r_1 \geq \cdots \geq r_m$, of $\tilde{M}'$, where $\tilde{y}'_2 = f - \tilde{a}'$, $\tilde{a}' = f(\tilde{p}')$, centered at $\tilde{p}'$, in which the tensor fields $\tilde{\Lambda}'_0$, $\tilde{N}'$ and $\tilde{T}'$ are written as their models (67), (68) and (74). An one-codimensional submanifold of $\tilde{M}'$, transverse to $\tilde{T}'$ and passing by $\tilde{p}'$, is the hypersurface Σ' of $\tilde{M}'$ defined by the equation $\tilde{x}_1'^1 = 0$, (it can also be seen as the hypersurface of level 2/3 of the functional coefficient $\tilde{x}_1'^1 + 2/3$ of $\partial/\partial\tilde{x}_1'^1$ in the considered model expression of $\tilde{T}'$). Moreover, a function a defined on a well chosen tubular neighbourhood $\tilde{U}'$ of Σ' in $\tilde{M}'$, which vanishes nowhere on $\tilde{U}'$, equal to 1 on Σ' and homogeneous of degree 1 with respect to $\tilde{T}'$, is the function

$$a((\tilde{x}_j'^i), \tilde{y}_1', \tilde{y}_2') = \tfrac{3}{2}\tilde{x}_1'^1 + 1.$$

We denote by $\pi' : \tilde{U}' \to \Sigma'$ the projection parallel to the integral curves of $\tilde{T}'$, by $T_{\Sigma'}\pi' : T_{\Sigma'}\tilde{U}' \to T\Sigma'$ the associated vector bundle projection of $T_{\Sigma'}\tilde{U}'$ onto its subbundle $T\Sigma'$, by ${}^{\mathrm{t}}T_{\Sigma'}\pi' : T^*\Sigma' \to T^*_{\Sigma'}\tilde{U}'$ the transpose of $T_{\Sigma'}\pi'$, and by $(T_{\Sigma'}\pi')_{\mathrm{h}}$ the restriction of $T_{\Sigma'}\pi'$ to the horizontal subbundle $T\Sigma'$ of $T_{\Sigma'}\tilde{U}'$, which is a bijection.

Let $((\Lambda'_{0\Sigma'}, E'_{0\Sigma'}), \mathcal{N}'_{\Sigma'})$, $\mathcal{N}'_{\Sigma'} := (N'_{\Sigma'}, Y'_{\Sigma'}, \gamma'_{\Sigma'}, g'_{\Sigma'})$, be the Jacobi–Nijenhuis structure induced on Σ' by the homogeneous symplectic Poisson–Nijenhuis structure $(\tilde{\Lambda}_0', \tilde{N}', \tilde{T}')$ of $\tilde{M}'$ (cf. Proposition 2.12). One has

$$\Lambda'^{\#}_{0\Sigma'} = T_{\Sigma'}\pi' \circ (a\tilde{\Lambda}'^{\#}_0)|_{\Sigma'} \circ {}^{\mathrm{t}}T_{\Sigma'}\pi', \tag{95}$$

$$E'_{0\Sigma'} = T_{\Sigma'}\pi'(\tilde{\Lambda}'^{\#}_0(da)|_{\Sigma'}), \tag{96}$$

$$N'_{\Sigma'} = T_{\Sigma'}\pi' \circ \tilde{N}'|_{\Sigma'} \circ (T_{\Sigma'}\pi')_{\mathrm{h}}^{-1}, \tag{97}$$

$$Y'_{\Sigma'} = T_{\Sigma'}\pi'((\tilde{N}'\tilde{T}')|_{\Sigma'}), \tag{98}$$

$$\gamma'_{\Sigma'} = ({}^{\mathrm{t}}\tilde{N}'da)\Big|_{\Sigma'} - \left\langle ({}^{\mathrm{t}}\tilde{N}'da)|_{\Sigma'}, \left.\frac{\partial}{\partial \tilde{x}_1'^1}\right|_{\Sigma'} \right\rangle d\tilde{x}_1'^1|_{\Sigma'}, \tag{99}$$

$$g'_{\Sigma'} = \langle da|_{\Sigma'}, (\tilde{N}'\tilde{T}')|_{\Sigma'}\rangle. \tag{100}$$

Their computation yields:

$$\begin{aligned}\Lambda'_{0\Sigma'} = {} & -\frac{3}{2}\left[\sum_{k=2}^{r_1}\tilde{x}_{2k-1}'^1\frac{\partial}{\partial\tilde{x}_{2k-1}'^1} + \sum_{i=2}^{m}\left(\sum_{k=1}^{r_i}\tilde{x}_{2k-1}'^i\frac{\partial}{\partial\tilde{x}_{2k-1}'^i}\right) + \tilde{y}_1'\frac{\partial}{\partial\tilde{y}_1'}\right] \\ & \wedge\frac{\partial}{\partial\tilde{x}_2'^1} + \sum_{k=2}^{r_1}\frac{\partial}{\partial\tilde{x}_{2k-1}'^1}\wedge\frac{\partial}{\partial\tilde{x}_{2k}'^1} + \sum_{i=2}^{m}\left(\sum_{k=1}^{r_i}\frac{\partial}{\partial\tilde{x}_{2k-1}'^i}\wedge\frac{\partial}{\partial\tilde{x}_{2k}'^i}\right) + \frac{\partial}{\partial\tilde{y}_1'}\wedge\frac{\partial}{\partial\tilde{y}_2'},\end{aligned} \tag{101}$$

$$E'_{0\Sigma'} = \frac{3}{2}\frac{\partial}{\partial\tilde{x}_2'^1}, \tag{102}$$

$$\begin{aligned}N'_{\Sigma'} = {} & -(\tilde{y}_2' + \tilde{a}')Id_{\Sigma'} - \frac{3}{2}T'_{\Sigma'}\otimes d\tilde{x}_3'^1 + H'_{\Sigma'} + \frac{\partial}{\partial\tilde{y}_1'}\otimes\alpha'_{\Sigma'} \\ & + \left(\frac{3}{2}T'_{\Sigma'} - Z'_{\Sigma'}\right)\otimes d\tilde{y}_2',\end{aligned} \tag{103}$$

where $-(3/2)T'_{\Sigma'}$ is the projection of $(\partial/\partial\tilde{x}'^{1}_{1})|_{\Sigma'}$ on $T\Sigma'$ parallel to $\tilde{T}'$,

$$T'_{\Sigma'} = \sum_{k=2}^{r_1} \tilde{x}'^{1}_{2k-1} \frac{\partial}{\partial \tilde{x}'^{1}_{2k-1}} + \sum_{i=2}^{m} \left(\sum_{k=1}^{r_i} \tilde{x}'^{i}_{2k-1} \frac{\partial}{\partial \tilde{x}'^{i}_{2k-1}} \right) + \tilde{y}'_1 \frac{\partial}{\partial \tilde{y}'_1}, \tag{104}$$

$$\begin{aligned} H'_{\Sigma'} = {}& \sum_{k=2}^{r_1-1} \left(\frac{\partial}{\partial \tilde{x}'^{1}_{2k-1}} \otimes d\tilde{x}'^{1}_{2k+1} \right) + \sum_{k=1}^{r_1-1} \left(\frac{\partial}{\partial \tilde{x}'^{1}_{2k+2}} \otimes d\tilde{x}'^{1}_{2k} \right) \\ & + \sum_{i=2}^{m} \left[\sum_{k=1}^{r_i-1} \left(\frac{\partial}{\partial \tilde{x}'^{i}_{2k-1}} \otimes d\tilde{x}'^{i}_{2k+1} + \frac{\partial}{\partial \tilde{x}'^{i}_{2k+2}} \otimes d\tilde{x}'^{i}_{2k} \right) \right], \end{aligned} \tag{105}$$

$$\begin{aligned} \alpha'_{\Sigma'} = {}& d\tilde{x}'^{1}_{2} + \sum_{k=2}^{r_1} \left[\left(k - \frac{1}{2}\right) \tilde{x}'^{1}_{2k} d\tilde{x}'^{1}_{2k-1} + \left(k + \frac{1}{2}\right) \tilde{x}'^{1}_{2k-1} d\tilde{x}'^{1}_{2k} \right] \\ & + \sum_{i=2}^{m} \left(\sum_{k=1}^{r_i} \left[\left(k - \frac{1}{2}\right) \tilde{x}'^{i}_{2k} d\tilde{x}'^{i}_{2k-1} + \left(k + \frac{1}{2}\right) \tilde{x}'^{i}_{2k-1} d\tilde{x}'^{i}_{2k} \right] \right), \end{aligned} \tag{106}$$

$$\begin{aligned} Z'_{\Sigma'} = {}& \sum_{k=2}^{r_1} \left[\left(k + \frac{1}{2}\right) \tilde{x}'^{1}_{2k-1} \frac{\partial}{\partial \tilde{x}'^{1}_{2k-1}} \right] - \sum_{k=1}^{r_1} \left[\left(k - \frac{1}{2}\right) \tilde{x}'^{1}_{2k} \frac{\partial}{\partial \tilde{x}'^{1}_{2k}} \right] \\ & + \sum_{i=2}^{m} \left(\sum_{k=1}^{r_i} \left[\left(k + \frac{1}{2}\right) \tilde{x}'^{i}_{2k-1} \frac{\partial}{\partial \tilde{x}'^{i}_{2k-1}} - \left(k - \frac{1}{2}\right) \tilde{x}'^{i}_{2k} \frac{\partial}{\partial \tilde{x}'^{i}_{2k}} \right] \right), \end{aligned} \tag{107}$$

$$\begin{aligned} Y'_{\Sigma'} = {}& \sum_{k=2}^{r_1-1} \left(\tilde{x}'^{1}_{2k+1} - \frac{3}{2} \tilde{x}'^{1}_{3} \tilde{x}'^{1}_{2k-1} \right) \frac{\partial}{\partial \tilde{x}'^{1}_{2k-1}} - \frac{3}{2} \tilde{x}'^{1}_{3} \tilde{x}'^{1}_{2r_1-1} \frac{\partial}{\partial \tilde{x}'^{1}_{2r_1-1}} \\ & + \sum_{i=2}^{m} \left[\sum_{k=1}^{r_i-1} \left(\tilde{x}'^{i}_{2k+1} - \frac{3}{2} \tilde{x}'^{1}_{3} \tilde{x}'^{i}_{2k-1} \right) \frac{\partial}{\partial \tilde{x}'^{i}_{2k-1}} \right] - \sum_{i=2}^{m} \frac{3}{2} \tilde{x}'^{1}_{3} \tilde{x}'^{i}_{2r_i-1} \frac{\partial}{\partial \tilde{x}'^{i}_{2r_i-1}} \\ & + \left(\frac{1}{3} \tilde{x}'^{1}_{2} + \sum_{k=2}^{r_1} \left(k - \frac{1}{2}\right) \tilde{x}'^{1}_{2k-1} \tilde{x}'^{1}_{2k} \right. \\ & \left. + \sum_{i=2}^{m} \left[\sum_{k=1}^{r_i} \left(k - \frac{1}{2}\right) \tilde{x}'^{i}_{2k-1} \tilde{x}'^{i}_{2k} \right] - \frac{3}{2} \tilde{x}'^{1}_{3} \tilde{y}'_1 \right) \frac{\partial}{\partial \tilde{y}'_1}, \end{aligned} \tag{108}$$

$$\gamma'_{\Sigma'} = \tfrac{3}{2}(d\tilde{x}'^{1}_{3} - d\tilde{y}'_2), \tag{109}$$

$$g'_{\Sigma'} = -(\tilde{y}'_2 + \tilde{a}') + \tfrac{3}{2}\tilde{x}'^{1}_{3}. \tag{110}$$

(Taking into account the remark of Theorem 3.1, if $df(\tilde{p}') = 0$, the obtained local expressions of the tensor fields of $((\Lambda'_{0\Sigma'}, E'_{0\Sigma'}), \mathcal{N}_{\Sigma'})$ do not include the $\tilde{y}'_1$ and $\tilde{y}'_2$ coordinates.)

Now, we consider a local coordinate system $\tilde{x}''$ of $\tilde{M}''$, centered at $\tilde{p}''$, in which $(\tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ has the expression of its model (see Theorem 3.3), and the product system $((\tilde{x}_j'^i), \tilde{y}_1', \tilde{y}_2'; \tilde{x}'')$, $i = 1, \dots, m$, $j = 1, \dots, 2r_i$, $r_1 \geq \cdots \geq r_m$, of $\tilde{M} = \tilde{M}' \times \tilde{M}''$, where $\tilde{y}_2' = f - \tilde{a}'$, $\tilde{a}' = f(\tilde{p}')$, centered at $\tilde{p} = (\tilde{p}', \tilde{p}'')$. Furthermore, we take the submanifold $\Sigma = \Sigma' \times \tilde{M}''$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$ of codimension 1, transverse to $\tilde{T}' + \tilde{T}''$, defined, of course, by $\tilde{x}_1'^1 = 0$.

Let $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, be the Jacobi–Nijenhuis structure induced on $\Sigma = \Sigma' \times \tilde{M}''$ by the homogeneous symplectic Poisson–Nijenhuis product structure $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T}) = (\tilde{\Lambda}_0', \tilde{N}', \tilde{T}') + (\tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$ (cf. Propositions 2.12 and 2.14). From Proposition 2.14, one has

$$\Lambda_{0\Sigma} = \Lambda_{0\Sigma'}' + \tilde{\Lambda}_0'' - \tilde{T}'' \wedge E_{0\Sigma'}' \quad \text{and} \quad E_{0\Sigma} = E_{0\Sigma'}', \tag{111}$$

$$N_\Sigma = N_{\Sigma'}' + \tilde{N}'' - \tilde{T}'' \otimes \gamma_{\Sigma'}', \tag{112}$$

$$Y_\Sigma = Y_{\Sigma'}' + \left(\tilde{N}'' - g_{\Sigma'}' Id_{T\tilde{M}''}\right) \tilde{T}'', \tag{113}$$

$$\gamma_\Sigma = \gamma_{\Sigma'}', \tag{114}$$

$$g_\Sigma = g_{\Sigma'}'. \tag{115}$$

The local expressions of the tensor fields $((\Lambda_{0\Sigma'}', E_{0\Sigma'}'), \mathcal{N}_{\Sigma'}')$, $\mathcal{N}_{\Sigma'}' := (N_{\Sigma'}', Y_{\Sigma'}', \gamma_{\Sigma'}', g_{\Sigma'}')$, in the coordinates of Σ', are given by Eqs. (101)–(110), and those of $(\tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$, in the considered coordinate system $\tilde{x}''$ of $\tilde{M}''$, are known by Theorem 3.3. Hence, formulæ (111)–(115) give us the local expression of the tensor fields of $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, in the coordinate product system $(\tilde{x}_2'^1, \dots, \tilde{x}_{2r_m}'^m, \tilde{y}_1', \tilde{y}_2'; \tilde{x}'')$ of $\Sigma = \Sigma' \times \tilde{M}''$.

4.1.2. *Study of Case 2*

We work as in Case 1. We denote by $(\tilde{M}', \tilde{\Lambda}_0', \tilde{N}', \tilde{T}')$ the factor of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ whose homothety vector field $\tilde{T}'$ is transverse to M at p, and we assume that its recursion operator $\tilde{N}'$ has a characteristic polynomial of type $\mathcal{P}_{\tilde{N}'}(\lambda) = (\lambda^2 + f\lambda + h)^q$, $q \leq n+1$, with $f^2 - 4h$ locally strictly negative. Then, on a neighbourhood of $\tilde{p}$ in $\tilde{M}$, $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T}) = (\tilde{M}', \tilde{\Lambda}_0', \tilde{N}', \tilde{T}') \times (\tilde{M}'', \tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$, where $(\tilde{M}'', \tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ is the product of the other factors of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. If $\tilde{p}'$ and $\tilde{p}''$ are, respectively, the projections of $\tilde{p}$ on $\tilde{M}'$ and $\tilde{M}''$, the normal form of $(\tilde{M}', \tilde{\Lambda}_0', \tilde{N}', \tilde{T}')$, on a neighbourhood of $\tilde{p}'$ in $\tilde{M}'$, is given by Theorem 3.2 and Eq. (77), and the one of $(\tilde{M}'', \tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$, on a neighbourhood of $\tilde{p}''$ in $\tilde{M}''$, by Theorem 3.3.

Let $((\tilde{x}_l'^j), \tilde{u}_1', \tilde{u}_2', (\tilde{y}_l'^j), \tilde{v}_1', \tilde{v}_2')$, $j = 1, \dots, m$, $l = 1, \dots, 2r_j$, $r_1 \geq \cdots \geq r_m$, be a local coordinate system of $\tilde{M}'$, centered at $\tilde{p}'$, in which the tensor fields $\tilde{\Lambda}_0'$, $\tilde{N}'$ and $\tilde{T}'$ are expressed as their models (Eq. (75)–(77)). To the role of an one-codimensional submanifold of $\tilde{M}'$ transverse to $\tilde{T}'$, we take the hypersurface Σ' of $\tilde{M}'$ through $\tilde{p}'$ that is defined by the equation $\tilde{x}_1'^1 = 0$. A function a defined on a well chosen tubular neighbourhood $\tilde{U}'$ of Σ' in $\tilde{M}'$, which never vanishes on $\tilde{U}'$, equal to 1 on Σ' and homogeneous of degree 1 with respect to $\tilde{T}'$, is the function

$$a((\tilde{x}_l'^j), \tilde{u}_1', \tilde{u}_2', (\tilde{y}_l'^j), \tilde{v}_1', \tilde{v}_2') = \tfrac{3}{2}\tilde{x}_1'^1 + 1.$$

We denote by $\pi' : \tilde{U}' \to \Sigma'$ the projection parallel to the integral curves of $\tilde{T}'$, by $T_{\Sigma'}\pi' : T_{\Sigma'}\tilde{U}' \to T\Sigma'$ the associated vector bundle projection of $T_{\Sigma'}\tilde{U}'$ onto its subbundle $T\Sigma'$, by ${}^{t}T_{\Sigma'}\pi' : T^*\Sigma' \to T^*_{\Sigma'}\tilde{U}'$ the transpose of $T_{\Sigma'}\pi'$, and by $(T_{\Sigma'}\pi')_{\mathrm{h}}$ the restriction of $T_{\Sigma'}\pi'$ to the horizontal subbundle $T\Sigma'$ of $T_{\Sigma'}\tilde{U}'$, which is a bijection.

Let $((\Lambda'_{0\Sigma'}, E'_{0\Sigma'}), \mathcal{N}'_{\Sigma'})$, $\mathcal{N}'_{\Sigma'} := (N'_{\Sigma'}, Y'_{\Sigma'}, \gamma'_{\Sigma'}, g'_{\Sigma'})$, be the Jacobi–Nijenhuis structure induced on Σ' by the homogeneous symplectic Poisson–Nijenhuis structure $(\tilde{\Lambda}_0', \tilde{N}', \tilde{T}')$ of $\tilde{M}'$ (cf. Proposition 2.12). The tensor fields defining this structure are given, respectively, by the formulæ (95)–(100). In this case, their computation yields

$$\begin{aligned}\Lambda'_{0\Sigma'} = {}& -\frac{3}{8}\Bigg[\sum_{k=2}^{r_1}\tilde{x}'^1_{2k-1}\frac{\partial}{\partial\tilde{x}'^1_{2k-1}} + \sum_{j=2}^{m}\left(\sum_{k=1}^{r_j}\tilde{x}'^j_{2k-1}\frac{\partial}{\partial\tilde{x}'^j_{2k-1}}\right)\\ &+ \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\tilde{y}'^j_{2k-1}\frac{\partial}{\partial\tilde{y}'^j_{2k-1}}\right) + \tilde{u}_1'\frac{\partial}{\partial\tilde{u}_1'} + \tilde{v}_1'\frac{\partial}{\partial\tilde{v}_1'}\Bigg]\\ &\wedge\frac{\partial}{\partial\tilde{x}'^1_2} + \sum_{k=2}^{r_1}\frac{1}{4}\frac{\partial}{\partial\tilde{x}'^1_{2k-1}}\wedge\frac{\partial}{\partial\tilde{x}'^1_{2k}} + \sum_{j=2}^{m}\left(\sum_{k=1}^{r_j}\frac{1}{4}\frac{\partial}{\partial\tilde{x}'^j_{2k-1}}\wedge\frac{\partial}{\partial\tilde{x}'^j_{2k}}\right)\\ &- \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\frac{1}{4}\frac{\partial}{\partial\tilde{y}'^j_{2k-1}}\wedge\frac{\partial}{\partial\tilde{y}'^j_{2k}}\right) + \frac{1}{4}\frac{\partial}{\partial\tilde{u}_1'}\wedge\frac{\partial}{\partial\tilde{u}_2'} - \frac{1}{4}\frac{\partial}{\partial\tilde{v}_1'}\wedge\frac{\partial}{\partial\tilde{v}_2'},\end{aligned}\tag{116}$$

$$E'_{0\Sigma'} = \frac{3}{8}\frac{\partial}{\partial\tilde{x}'^1_2},\tag{117}$$

$$\begin{aligned}N'_{\Sigma'} = {}& -(\tilde{u}_2' + \tilde{a}')Id_{\Sigma'} - \frac{3}{2}T'_{\Sigma'}\otimes d\tilde{x}'^1_3 + H'_{\tilde{x}'\Sigma'} - (\tilde{v}_2' + \tilde{b}')J_{\Sigma'}\\ &- (\tilde{v}_2' + \tilde{b}')\frac{3}{2}T'_{\Sigma'}\otimes d\tilde{y}'^1_1 + \left(\frac{3}{2}T'_{\Sigma'} - Z'_{\tilde{x}'\Sigma'}\right)\otimes(d\tilde{u}_2' + d\tilde{v}_2') + H'_{\tilde{y}'}\\ &- Z'_{\tilde{y}'}\otimes(d\tilde{u}_2' - d\tilde{v}_2') + \frac{\partial}{\partial\tilde{u}_1'}\otimes(\alpha'_{\tilde{x}'\Sigma'} - \alpha'_{\tilde{y}'}) + \frac{\partial}{\partial\tilde{v}_1'}\otimes(\alpha'_{\tilde{x}'\Sigma'} + \alpha'_{\tilde{y}'}),\end{aligned}\tag{118}$$

where $-(3/2)T'_{\Sigma'}$ is the projection of $(\partial/\partial\tilde{x}'^1_1)|_{\Sigma'}$ on $T\Sigma'$ in the direction of $\tilde{T}'$,

$$\begin{aligned}T'_{\Sigma'} = {}& \sum_{k=2}^{r_1}\tilde{x}'^1_{2k-1}\frac{\partial}{\partial\tilde{x}'^1_{2k-1}} + \sum_{j=2}^{m}\left(\sum_{k=1}^{r_j}\tilde{x}'^j_{2k-1}\frac{\partial}{\partial\tilde{x}'^j_{2k-1}}\right) + \sum_{j=1}^{m}\left(\sum_{k=1}^{r_j}\tilde{y}'^j_{2k-1}\frac{\partial}{\partial\tilde{y}'^j_{2k-1}}\right)\\ &+ \tilde{u}_1'\frac{\partial}{\partial\tilde{u}_1'} + \tilde{v}_1'\frac{\partial}{\partial\tilde{v}_1'},\end{aligned}$$

$$J_{\Sigma'} = \sum_{l=2}^{2r_1}\left(\frac{\partial}{\partial \tilde{y}_l'^1}\otimes d\tilde{x}_l'^1 - \frac{\partial}{\partial \tilde{x}_l'^1}\otimes d\tilde{y}_l'^1\right) + \sum_{j=2}^{m}\sum_{l=1}^{2r_j}\left(\frac{\partial}{\partial \tilde{y}_l'^j}\otimes d\tilde{x}_l'^j - \frac{\partial}{\partial \tilde{x}_l'^j}\otimes d\tilde{y}_l'^j\right)$$
$$- \frac{\partial}{\partial \tilde{u}_1'}\otimes d\tilde{v}_1' - \frac{\partial}{\partial \tilde{u}_2'}\otimes d\tilde{v}_2' + \frac{\partial}{\partial \tilde{v}_1'}\otimes d\tilde{u}_1' + \frac{\partial}{\partial \tilde{v}_2'}\otimes d\tilde{u}_2',$$

the tensor fields $H'_{\tilde{x}'\Sigma'}, \alpha'_{\tilde{x}'\Sigma'}, Z'_{\tilde{x}'\Sigma'}$ have, respectively, the expressions (105)–(107), and $H'_{\tilde{y}'}, \alpha'_{\tilde{y}'}, Z'_{\tilde{y}'}$ those that appear in Theorem 3.2,

$$\begin{aligned}
Y'_{\Sigma'} = {} & \sum_{k=2}^{r_1-1}\left[\tilde{x}_{2k+1}'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_{2k-1}'^1 - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{x}_{2k-1}'^1\right]\frac{\partial}{\partial \tilde{x}_{2k-1}'^1} \\
& + \sum_{j=2}^{m}\sum_{k=1}^{r_j-1}\left[\tilde{x}_{2k+1}'^j + (\tilde{v}_2' + \tilde{b}')\tilde{y}_{2k-1}'^j - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{x}_{2k-1}'^j\right]\frac{\partial}{\partial \tilde{x}_{2k-1}'^j} \\
& + \sum_{j=1}^{m}\left[(\tilde{v}_2' + \tilde{b}')\tilde{y}_{2r_j-1}'^j - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{x}_{2r_j-1}'^j\right]\frac{\partial}{\partial \tilde{x}_{2r_j-1}'^j} \\
& + \left[-\frac{2}{3}(\tilde{v}_2' + \tilde{b}') + \tilde{y}_3'^1 - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{y}_1'^1\right]\frac{\partial}{\partial \tilde{y}_1'^1} \\
& + \sum_{k=2}^{r_1-1}\left[-(\tilde{v}_2' + \tilde{b}')\tilde{x}_{2k-1}'^1 + \tilde{y}_{2k+1}'^1 - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{y}_{2k-1}'^1\right]\frac{\partial}{\partial \tilde{y}_{2k-1}'^1} \\
& + \sum_{j=2}^{m}\sum_{k=1}^{r_j-1}\left[-(\tilde{v}_2' + \tilde{b}')\tilde{x}_{2k-1}'^j + \tilde{y}_{2k+1}'^j - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{y}_{2k-1}'^j\right]\frac{\partial}{\partial \tilde{y}_{2k-1}'^j} \\
& + \sum_{j=1}^{m}\left[-(\tilde{v}_2' + \tilde{b}')\tilde{x}_{2r_j-1}'^j - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{y}_{2r_j-1}'^j\right]\frac{\partial}{\partial \tilde{y}_{2r_j-1}'^j} \\
& + \left[\frac{1}{3}\tilde{x}_2'^1 + \sum_{k=2}^{r_1}\left(k - \frac{1}{2}\right)\tilde{x}_{2k-1}'^1\tilde{x}_{2k}'^1 + \sum_{j=2}^{m}\sum_{k=1}^{r_j}\left(k - \frac{1}{2}\right)\tilde{x}_{2k-1}'^j\tilde{x}_{2k}'^j\right. \\
& \left. - \sum_{j=1}^{m}\sum_{k=1}^{r_j}\left(k - \frac{1}{2}\right)\tilde{y}_{2k-1}'^j\tilde{y}_{2k}'^j + \tilde{v}_1'(\tilde{v}_2' + \tilde{b}') - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{u}'_1\right]\frac{\partial}{\partial \tilde{u}_1'} \\
& + \left[\frac{1}{3}\tilde{x}_2'^1 + \sum_{k=2}^{r_1}\left(k - \frac{1}{2}\right)\tilde{x}_{2k-1}'^1\tilde{x}_{2k}'^1 + \sum_{j=2}^{m}\sum_{k=1}^{r_j}\left(k - \frac{1}{2}\right)\tilde{x}_{2k-1}'^j\tilde{x}_{2k}'^j\right. \\
& \left. + \sum_{j=1}^{m}\sum_{k=1}^{r_j}\left(k - \frac{1}{2}\right)\tilde{y}_{2k-1}'^j\tilde{y}_{2k}'^j - \tilde{u}_1'(\tilde{v}_2' + \tilde{b}') - \frac{3}{2}(\tilde{x}_3'^1 + (\tilde{v}_2' + \tilde{b}')\tilde{y}_1'^1)\tilde{v}_1'\right]\frac{\partial}{\partial \tilde{v}_1'},
\end{aligned} \tag{119}$$

$$\gamma'_{\Sigma'} = \tfrac{3}{2}(d\tilde{x}'^1_3 + (\tilde{v}'_2 + \tilde{b}')d\tilde{y}'^1_1 - d\tilde{u}'_2 - d\tilde{v}'_2), \tag{120}$$

$$g'_{\Sigma'} = -(\tilde{u}'_2 + \tilde{a}') + \tfrac{3}{2}(\tilde{x}'^1_3 + (\tilde{v}'_2 + \tilde{b}')\tilde{y}'^1_1). \tag{121}$$

Afterwards, we consider a local coordinate system $\tilde{x}''$ of $\tilde{M}''$, centered at $\tilde{p}''$, in which $(\tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ has the expression of its model (see Theorem 3.3), and also the product system $((\tilde{x}'^j_l), \tilde{u}'_1, \tilde{u}'_2, (\tilde{y}'^j_l), \tilde{v}'_1, \tilde{v}'_2; \tilde{x}'')$, $j = 1, \dots, m$, $l = 1, \dots, 2r_j$, $r_1 \geq \cdots \geq r_m$, of $\tilde{M} = \tilde{M}' \times \tilde{M}''$, centered at $\tilde{p} = (\tilde{p}', \tilde{p}'')$. Moreover, we take the submanifold $\Sigma = \Sigma' \times \tilde{M}''$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$ of codimension 1, transverse to $\tilde{T}' + \tilde{T}''$, defined, of course, by $\tilde{x}'^1_1 = 0$.

Let $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, be the Jacobi–Nijenhuis structure induced on $\Sigma = \Sigma' \times \tilde{M}''$ by the homogeneous symplectic Poisson–Nijenhuis product structure $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T}) = (\tilde{\Lambda}'_0, \tilde{N}', \tilde{T}') + (\tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$, (cf. Propositions 2.12 and 2.14). From Proposition 2.14 we deduce the expressions of the tensor fields of $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and of $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$ that are represented, respectively, by formulæ (111) and (112)–(115). Then, taking into account the already established local expressions of $(\Lambda'_{0\Sigma'}, E'_{0\Sigma'})$ and of $\mathcal{N}'_{\Sigma'} := (N'_{\Sigma'}, Y'_{\Sigma'}, \gamma'_{\Sigma'}, g'_{\Sigma'})$ in the coordinates of Σ' (see relations (116)–(121)), and also the local expressions of $(\tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ in the considered coordinate system $\tilde{x}''$ of $\tilde{M}''$ (cf. Theorem 3.3), from Eqs. (111)–(115) we may deduce the local expressions of $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and of $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$ in the coordinate product system $(\tilde{x}'^1_2, \dots, \tilde{x}'^m_{2r_m}, \tilde{u}'_1, \tilde{u}'_2, \tilde{y}'^1_1, \dots, \tilde{y}'^m_{2r_m}, \tilde{v}'_1, \tilde{v}'_2; \tilde{x}'')$ of $\Sigma = \Sigma' \times \tilde{M}''$.

In conclusion, we present the following theorem.

Theorem 4.1. *Let* $((\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, *be a transitive Jacobi–Nijenhuis structure defined on a* $(2n+1)$*-dimensional differentiable manifold* M, $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ *the associated homogeneous symplectic Poisson–Nijenhuis structure on* $\tilde{M} = M \times \boldsymbol{R}$, *and* p *a generic point of* M, *viewed as the projection on* M *of a regular point* $\tilde{p}$ *of* $\tilde{M}$ *with respect to* $\tilde{N}$. *Also, let* $(\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$ *be a factor of the "model decomposition" of* $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ *whose homothety vector field* $\tilde{T}'$ *is supposed to be transverse to* M *at* p, Σ *a submanifold of* $\tilde{M}$ *through* $\tilde{p}$ *of codimension 1 and transverse to* $\tilde{T}$, *and* $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, *the Jacobi–Nijenhuis structure induced on* Σ *by* $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. *If the characteristic polynomial of* $\tilde{N}'$ *is of the type* $\mathcal{P}_{\tilde{N}'}(\lambda) = (\lambda + f)^{2q}$ (*respectively* $\mathcal{P}_{\tilde{N}'}(\lambda) = (\lambda^2 + f\lambda + h)^q$, *with* $f^2 - 4h$ *locally strictly negative*), $q \leq n+1$, *then, there exists a neighbourhood of* $\tilde{p}$ *in* Σ *with a coordinates system, centered at* p, *in which the tensor fields of* $(\Lambda_{0\Sigma}, E_{0\Sigma})$ *and of* $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$ *are written, respectively, as* Eqs. (111) *and* (112)–(115), *taking into account formulæ* (101)–(110) (*respectively* (116)–(121)). *The structure* $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$ *is locally equivalent to a conformal structure to* $((\Lambda_0, E_0), \mathcal{N})$.

4.2. Local models of even-dimensional Jacobi–Nijenhuis manifolds

Let $((\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi–Nijenhuis structure defined on a $2n$-dimensional differentiable manifold M and $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ the associated homogeneous Poisson–Nijenhuis structure defined on $\tilde{M} = M \times \boldsymbol{R}$ (cf. Proposition 2.16). We assume that the Poissonization $\tilde{\Lambda}_0 = e^{-t}(\Lambda_0 + (\partial/\partial t) \wedge E_0)$ (t is the canonical coordinate on the factor $\boldsymbol{R}$) of (Λ_0, E_0) is of maximum rank on an open dense subset of $\tilde{M} = M \times \boldsymbol{R}$. Let p be a generic point of M, i.e. p can be viewed as the projection on M of a regular point $\tilde{p}$ of $\tilde{M}$, with respect to $\tilde{N} = N + Y \otimes dt + (\partial/\partial t) \otimes \gamma + g(\partial/\partial t) \otimes dt$, such that corank $\tilde{\Lambda}_0(\tilde{p}) = 1$. Our aim is to construct a model of $((\Lambda_0, E_0), \mathcal{N})$ on a neighbourhood of p. We remark that the characteristic leaf C_0 of (Λ_0, E_0) through p is the projection on M, parallel to the integral curves of $\tilde{T} = (\partial/\partial t)$, of the symplectic leaf $\tilde{S}_0$ of $\tilde{\Lambda}_0$ through $\tilde{p}$ (see Section 2.2); of course, $\dim \tilde{S}_0 = 2n$. Then,

- if $\tilde{T} = (\partial/\partial t)$ is tangent to $\tilde{S}_0$, C_0 has dimension $2n - 1$, and we have that rank $\Lambda_0(p) = 2n - 2$ and $E_0(p) \notin \Im\, \Lambda_0^{\#}(p)$;
- if $\tilde{T} = (\partial/\partial t)$ is not tangent to $\tilde{S}_0$, C_0 has dimension $2n$, i.e. $\dim C_0 = \dim M$, consequently rank $\Lambda_0(p) = 2n$ and $E_0(p) \in \Im\, \Lambda_0^{\#}(p)$, and the restriction to $\tilde{S}_0$ of the projection of $\tilde{M}$ on M parallel to the integral curves of $\tilde{T} = (\partial/\partial t)$ is a local diffeomorphism of $\tilde{S}_0$ onto C_0. Then, in this case, (Λ_0, E_0) is transitive on a neighbourhood of p in M.

Hence, in order to establish a model of $((\Lambda_0, E_0), \mathcal{N})$ on a neighbourhood of p, we will study separately the above mentioned cases.

4.2.1. *Study of the case where $\partial/\partial t$ is tangent to $\tilde{S}_0$*

In this case, for the construction of a normal form of $((\Lambda_0, E_0), \mathcal{N})$ on a neighbourhood of p, we apply the technique developed in the previous paragraph. From Theorem 3.4 and the study that follows, on a neighbourhood of $\tilde{p}$ in $\tilde{M}$, the model of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ is a product of a homogeneous Poisson–Nijenhuis manifold $(\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$ of odd dimension $2l - 1$, $l \leq n + 1$, whose recursion operator $\tilde{N}'$ has a characteristic polynomial of type $\mathcal{P}_{\tilde{N}'}(\lambda) = (\lambda + f)^{2l-1}$ and whose homothety vector field $\tilde{T}'$ is tangent to the symplectic leaf $\tilde{S}'_0$ of $\tilde{\Lambda}'_0$ passing by the projection $\tilde{p}'$ of $\tilde{p}$ on $\tilde{M}'$, and a homogeneous symplectic Poisson–Nijenhuis manifold $(\tilde{M}'', \tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$. The normal form of $(\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$ is well described by Theorem 3.4 and Eq. (94) and the one of $(\tilde{M}'', \tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ by Theorem 3.3. In what follows, this decomposition of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ will be referred as the *"model decomposition"* of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. Since $\tilde{T} = \partial/\partial t$ is supposed to be transverse to M at p, we have that at least one of its components is transverse to M at p. We distinguish and we treat separately the following cases:

1. The component of $\tilde{T}$ that is transverse to M at p is $\tilde{T}'$.
2. The component of $\tilde{T}$ that is transverse to M at p is $\tilde{T}''$.

Case 1. We take the factor $(\tilde{M}', \tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$ of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ that possesses the properties stated above and whose homothety vector field $\tilde{T}'$ is supposed to be transverse to M at p. We assume that $df(\tilde{p}') \neq 0$, and we consider a local coordinate system $((\tilde{x}'^i_j), \tilde{y}')$, $i = 1, \ldots, m$, $j = 1, \ldots, 2r_i$, $r_1 \geq \cdots \geq r_m$, of $\tilde{M}'$, where $\tilde{y}' = f - \tilde{a}'$, $\tilde{a}' = f(\tilde{p}')$, centered at $\tilde{p}'$, in which the tensor fields $\tilde{\Lambda}'_0$, $\tilde{N}'$ and $\tilde{T}'$ are written, respectively, as their models (78), (79) and (94). For the role of an one-codimensional submanifold of $\tilde{M}'$ transverse to $\tilde{T}'$, we take the hypersurface Σ' of $\tilde{M}'$ defined by the equation $\tilde{x}'^1_1 = 0$; of course $\tilde{p}' \in \Sigma'$. A function a defined on a well chosen tubular neighbourhood $\tilde{U}'$ of Σ'

in $\tilde{M}'$, which never vanishes on $\tilde{U}'$, equal to 1 on Σ' and homogeneous of degree 1 with respect to $\tilde{T}'$, is the function

$$a((\tilde{x}_j'^i), \tilde{y}') = \tfrac{3}{2}\tilde{x}_1'^1 + 1.$$

Let $((\Lambda'_{0\Sigma'}, E'_{0\Sigma'}), \mathcal{N}'_{\Sigma'}), \mathcal{N}'_{\Sigma'} := (N'_{\Sigma'}, Y'_{\Sigma'}, \gamma'_{\Sigma'}, g'_{\Sigma'})$, be the Jacobi–Nijenhuis structure induced on Σ' by the homogeneous Poisson–Nijenhuis structure $(\tilde{\Lambda}'_0, \tilde{N}', \tilde{T}')$ of $\tilde{M}'$ (cf. Proposition 2.12). Developing the same reasoning as in Section 4.1, we obtain

$$\Lambda'_{0\Sigma'} = -\frac{3}{2}\left[\sum_{k=2}^{r_1}\tilde{x}_{2k-1}'^1\frac{\partial}{\partial\tilde{x}_{2k-1}'^1} + \sum_{i=2}^{m}\left(\sum_{k=1}^{r_i}\tilde{x}_{2k-1}'^i\frac{\partial}{\partial\tilde{x}_{2k-1}'^i}\right)\right] \wedge\frac{\partial}{\partial\tilde{x}_2'^1} + \sum_{k=2}^{r_1}\frac{\partial}{\partial\tilde{x}_{2k-1}'^1}\wedge\frac{\partial}{\partial\tilde{x}_{2k}'^1} + \sum_{i=2}^{m}\left(\sum_{k=1}^{r_i}\frac{\partial}{\partial\tilde{x}_{2k-1}'^i}\wedge\frac{\partial}{\partial\tilde{x}_{2k}'^i}\right), \tag{122}$$

$$E'_{0\Sigma'} = \frac{3}{2}\frac{\partial}{\partial\tilde{x}_2'^1}, \tag{123}$$

$$N'_{\Sigma'} = -(\tilde{y}' + \tilde{a}')Id_{\Sigma'} - \frac{3}{2}T'_{\Sigma'}\otimes d\tilde{x}_3'^1 + H'_{\Sigma'} + \left(\frac{3}{2}T'_{\Sigma'} - Z'_{\Sigma'}\right)\otimes d\tilde{y}', \tag{124}$$

where $-(3/2)T'_{\Sigma'}$ is the projection of $\partial/\partial\tilde{x}_1'^1|_{\Sigma'}$ on $T\Sigma'$ parallel to $\tilde{T}'$,

$$T'_{\Sigma'} = \sum_{k=2}^{r_1}\tilde{x}_{2k-1}'^1\frac{\partial}{\partial\tilde{x}_{2k-1}'^1} + \sum_{i=2}^{m}\left(\sum_{k=1}^{r_i}\tilde{x}_{2k-1}'^i\frac{\partial}{\partial\tilde{x}_{2k-1}'^i}\right),$$

and $H'_{\Sigma'}, Z'_{\Sigma'}$ are given, respectively, by Eqs. (105) and (107),

$$Y'_{\Sigma'} = \sum_{k=2}^{r_1-1}\left(\tilde{x}_{2k+1}'^1 - \frac{3}{2}\tilde{x}_3'^1\tilde{x}_{2k-1}'^1\right)\frac{\partial}{\partial\tilde{x}_{2k-1}'^1} - \frac{3}{2}\tilde{x}_3'^1\tilde{x}_{2r_1-1}'^1\frac{\partial}{\partial\tilde{x}_{2r_1-1}'^1} + \sum_{i=2}^{m}\left[\sum_{k=1}^{r_i-1}\left(\tilde{x}_{2k+1}'^i - \frac{3}{2}\tilde{x}_3'^1\tilde{x}_{2k-1}'^i\right)\frac{\partial}{\partial\tilde{x}_{2k-1}'^i}\right] - \sum_{i=2}^{m}\frac{3}{2}\tilde{x}_3'^1\tilde{x}_{2r_i-1}'^i\frac{\partial}{\partial\tilde{x}_{2r_i-1}'^i}, \tag{125}$$

$$\gamma'_{\Sigma'} = \tfrac{3}{2}(d\tilde{x}_3'^1 - d\tilde{y}'), \tag{126}$$

$$g'_{\Sigma'} = -(\tilde{y}' + \tilde{a}') + \tfrac{3}{2}\tilde{x}_3'^1. \tag{127}$$

(If $df(\tilde{p}') = 0$, the obtained local expressions of the tensor fields of the structure $((\Lambda'_{0\Sigma'}, E'_{0\Sigma'}), \mathcal{N}'_{\Sigma'})$ do not include the $\tilde{x}_{2r_m}'^m$ and $\tilde{y}'$ coordinates.)

Now, we consider a local coordinate system $\tilde{x}''$ of $\tilde{M}''$, centered at $\tilde{p}''$ (we denote by $\tilde{p}''$ the projection of $\tilde{p}$ on $\tilde{M}''$), in which $(\tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ has the expression of its model presented by Theorem 3.3, and also the product system $((\tilde{x}_j'^i), \tilde{y}'; \tilde{x}'')$, $i = 1, \dots, m$, $j = 1, \dots, 2r_i$, $r_1 \geq \dots \geq r_m$, of $\tilde{M} = \tilde{M}' \times \tilde{M}''$, where $\tilde{y}' = f - \tilde{a}'$, $\tilde{a}' = f(\tilde{p}')$, centered at $\tilde{p} = (\tilde{p}', \tilde{p}'')$.

Moreover, we consider the hypersurface $\Sigma = \Sigma' \times \tilde{M}''$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$ defined by the equation $\tilde{x}_1'^1 = 0$. Of course, it is an one-codimensional submanifold of $\tilde{M} = \tilde{M}' \times \tilde{M}'$, passing by $\tilde{p}$, transverse to the homothety vector field $\tilde{T}' + \tilde{T}''$.

Let $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, be the Jacobi–Nijenhuis structure induced on $\Sigma = \Sigma' \times \tilde{M}''$ by the homogeneous Poisson–Nijenhuis product structure $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T}) = (\tilde{\Lambda}_0', \tilde{N}', \tilde{T}') + (\tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$, (cf. Propositions 2.12 and 2.14). From Proposition 2.14, we deduce the expressions (111)–(115) of the tensor fields of $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$. Since we know the local models of $((\Lambda'_{0\Sigma'}, E'_{0\Sigma'}), \mathcal{N}'_{\Sigma'})$, $\mathcal{N}'_{\Sigma'} := (N'_{\Sigma'}, Y'_{\Sigma'}, \gamma'_{\Sigma'}, g'_{\Sigma'})$, in the coordinates $(\tilde{x}_2'^1, \ldots, \tilde{x}_{2r_m}'^m, \tilde{y}')$ of Σ' (cf. relations (122)–(127)), and of $(\tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ in the considered coordinate system $\tilde{x}''$ of $\tilde{M}''$ (cf. Theorem 3.3), (111)–(115) give us the local writing of $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, in the local coordinate product system $(\tilde{x}_2'^1, \ldots, \tilde{x}_{2r_m}'^m, \tilde{y}'; \tilde{x}'')$ of $\Sigma = \Sigma' \times \tilde{M}''$.

Then, we are lead to the following theorem.

Theorem 4.2. *Let $((\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi–Nijenhuis structure defined on a 2n-dimensional differentiable manifold M and $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ the associated homogeneous Poisson–Nijenhuis structure on $\tilde{M} = M \times \boldsymbol{R}$. Suppose that (Λ_0, E_0) is such that its Poissonization $\tilde{\Lambda}_0$ is of maximum rank on an open dense subset of $\tilde{M} = M \times \boldsymbol{R}$. Let p be a generic point of M, viewed as the projection on M of a regular point $\tilde{p} \in \tilde{M}$, with respect to $\tilde{N}$, such that corank $\tilde{\Lambda}_0(\tilde{p}) = 1$, and let $\tilde{S}_0$ be the symplectic leaf of $\tilde{\Lambda}_0$ through $\tilde{p}$. Also let $(\tilde{M}', \tilde{\Lambda}_0', \tilde{N}', \tilde{T}')$ be the odd-dimensional factor of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ whose homothety vector field $\tilde{T}'$ is assumed to be transverse to M at p, Σ an one-codimensional submanifold of $\tilde{M}$, passing by $\tilde{p}$, transverse to $\tilde{T}$, and $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, the Jacobi–Nijenhuis structure induced on Σ by $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. If $\tilde{T}$ is tangent to $\tilde{S}_0$, then, there exists a neighbourhood of $\tilde{p}$ in Σ with a system of coordinates, centered at $\tilde{p}$, in which the tensor fields of $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and of $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$ are written, respectively, as* Eqs. (111) *and* (112)–(115) (*taking into account* (122)–(127)). *The structure $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$ is locally equivalent to a conformal structure to $((\Lambda_0, E_0), \mathcal{N})$.*

Case 2. Take the factor $(\tilde{M}'', \tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ of the "model decomposition" of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ which is a homogeneous symplectic Poisson–Nijenhuis manifold whose homothety vector field $\tilde{T}''$ is supposed to be transverse to M at p. Let $\tilde{p}''$ be the projection of $\tilde{p}$ on $\tilde{M}''$. From Theorem 3.3, on a neighbourhood of $\tilde{p}''$ in $\tilde{M}''$, $(\tilde{M}'', \tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ is identified with a finite product of homogenous symplectic Poisson–Nijenhuis manifolds whose recursion operator has a characteristic polynomial that is a power of an irreducible polynomial. Since $\tilde{T}''$ is transverse to M at p, at least one of its components, in the considered decomposition, is transverse to M at p.

Let Σ'' be a submanifold of $\tilde{M}''$ of codimension 1, passing by $\tilde{p}''$ and transverse to $\tilde{T}''$, and $((\Lambda''_{0\Sigma''}, E''_{0\Sigma''}), \mathcal{N}''_{\Sigma''})$, $\mathcal{N}''_{\Sigma''} := (N''_{\Sigma''}, Y''_{\Sigma''}, \gamma''_{\Sigma''}, g''_{\Sigma''})$, the Jacobi–Nijenhuis structure induced on Σ'' by the homogeneous symplectic Poisson–Nijenhuis structure $(\tilde{\Lambda}_0'', \tilde{N}'', \tilde{T}'')$ of $\tilde{M}''$ (cf. Proposition 2.12). The local model of $((\Lambda''_{0\Sigma''}, E''_{0\Sigma''}), \mathcal{N}''_{\Sigma''})$ is well known from Theorem 4.1.

Now, we consider the submanifold $\Sigma = \tilde{M}' \times \Sigma''$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$, which is, of course, one-codimensional and transverse to $\tilde{T}' + \tilde{T}''$. Let $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, be the Jacobi–Nijenhuis structure induced on $\Sigma = \tilde{M}' \times \Sigma''$ by the homogeneous Poisson–Nijenhuis product structure $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T}) = (\tilde{\Lambda}'_0, \tilde{N}', \tilde{T}') + (\tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ of $\tilde{M} = \tilde{M}' \times \tilde{M}''$ (cf. Propositions 2.12 and 2.14). From Proposition 2.14,

$$\Lambda_{0\Sigma} = \tilde{\Lambda}'_0 + \Lambda''_{0\Sigma''} - \tilde{T}' \wedge E''_{0\Sigma''} \quad \text{and} \quad E_{0\Sigma} = E''_{0\Sigma''}, \tag{128}$$

$$N_\Sigma = \tilde{N}' + N''_{\Sigma''} - \tilde{T}' \otimes \gamma''_{\Sigma''}, \tag{129}$$

$$Y_\Sigma = (\tilde{N}' - g''_{\Sigma''} Id_{T\tilde{M}'})\tilde{T}' + Y''_{\Sigma''}, \tag{130}$$

$$\gamma_\Sigma = \gamma''_{\Sigma''}, \tag{131}$$

$$g_\Sigma = g''_{\Sigma''}. \tag{132}$$

Then, if $\tilde{x}'$ is a local coordinate system of $\tilde{M}'$, centered at $\tilde{p}'$, in which the tensor fields $\tilde{\Lambda}'_0$, $\tilde{N}'$ and $\tilde{T}'$ are written, respectively, as Eqs. (78), (79) and (94), and if $\tilde{x}''_{\Sigma''}$ is a local coordinate system of Σ'', centered at $\tilde{p}''$, in which the tensor fields of $((\Lambda''_{0\Sigma''}, E''_{0\Sigma''}), \mathcal{N}''_{\Sigma''})$, $\mathcal{N}''_{\Sigma''} := (N''_{\Sigma''}, Y''_{\Sigma''}, \gamma''_{\Sigma''}, g''_{\Sigma''})$, have the expressions of their models (cf. Theorem 4.1), formulæ (128)–(132) give us the local expression of $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, in the local coordinate product system $(\tilde{x}'; \tilde{x}''_{\Sigma''})$ of $\Sigma = \tilde{M}' \times \Sigma''$.

So, we get the following theorem.

Theorem 4.3. *Let $((\Lambda_0, E_0), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi–Nijenhuis structure defined on a 2n-dimensional differentiable manifold M and $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ the associated homogeneous Poisson–Nijenhuis structure on $\tilde{M} = M \times \boldsymbol{R}$. Suppose that (Λ_0, E_0) is such that its Poissonization $\tilde{\Lambda}_0$ is of maximum rank on an open dense subset of $\tilde{M} = M \times \boldsymbol{R}$. Let p be a generic point of M, viewed as the projection on M of a regular point $\tilde{p} \in \tilde{M}$, with respect to $\tilde{N}$, such that* corank $\tilde{\Lambda}_0(\tilde{p}) = 1$, *and let $\tilde{S}_0$ be the symplectic leaf of $\tilde{\Lambda}_0$ through $\tilde{p}$. Also, let $(\tilde{M}'', \tilde{\Lambda}''_0, \tilde{N}'', \tilde{T}'')$ be the homogeneous symplectic Poisson–Nijenhuis manifold of the “model decomposition” of $(\tilde{M}, \tilde{\Lambda}_0, \tilde{N}, \tilde{T})$ whose homothety vector field $\tilde{T}''$ is supposed to be transverse to M at p, Σ an one-codimensional submanifold of $\tilde{M}$, passing by $\tilde{p}$, transverse to $\tilde{T}$, and $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$, $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$, the Jacobi–Nijenhuis structure induced on Σ by $(\tilde{\Lambda}_0, \tilde{N}, \tilde{T})$. If $\tilde{T}$ is tangent to $\tilde{S}_0$, then, there exists a neighbourhood of $\tilde{p}$ in Σ with a system of coordinates, centered at $\tilde{p}$, in which the tensor fields of $(\Lambda_{0\Sigma}, E_{0\Sigma})$ and of $\mathcal{N}_\Sigma := (N_\Sigma, Y_\Sigma, \gamma_\Sigma, g_\Sigma)$ are written, respectively, as Eq. (128) and (129)–(132) (taking into account the model expression of $((\Lambda''_{0\Sigma''}, E''_{0\Sigma''}), \mathcal{N}''_{\Sigma''})$ presented by Theorem 4.1). The structure $((\Lambda_{0\Sigma}, E_{0\Sigma}), \mathcal{N}_\Sigma)$ is locally equivalent to a conformal structure to $((\Lambda_0, E_0), \mathcal{N})$.*

4.2.2. Study of the case where $\partial/\partial t$ is not tangent to $\tilde{S}_0$

Consider the same context as in the beginning of Section 4.2 and assume that the homothety vector field $\tilde{T} = \partial/\partial t$ of $(\tilde{\Lambda}_0, \tilde{N})$ is not tangent to the symplectic leaf $\tilde{S}_0$ of $\tilde{\Lambda}_0$ through $\tilde{p}$. As we have remarked, in this case (Λ_0, E_0) is transitive on a neighbourhood

U of p in M. Then, there exists a differentiable function $f \in C^\infty(U, \boldsymbol{R})$ that vanishes nowhere on U such that the Jacobi structure (Λ_0^f, E_0^f), f-conformal to (Λ_0, E_0), is a symplectic Poisson structure on U, i.e. $\Lambda_0^f = f\Lambda_0$ is a nondegenerate Poisson tensor on U and $E_0^f = \Lambda_0^\#(df) + fE_0 = 0$ (cf. [11,2,9]).

Let $((\Lambda_0^f, E_0^f), \mathcal{N}^f)$, $\mathcal{N}^f := (N^f, Y^f, \gamma^f, g^f)$, be the Jacobi–Nijenhuis structure, f-conformal to $((\Lambda_0, E_0), \mathcal{N})$, and (Λ_1^f, E_1^f) the Jacobi structure, f-conformal to (Λ_1, E_1), $(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda_0, E_0)^\#$. From Proposition 2.11,

$$(\Lambda_1^f, E_1^f)^\# = \mathcal{N}^f \circ (\Lambda_0^f, E_0^f)^\#.$$

Then,

$$E_1^f = N^f E_0^f = 0,$$

(cf. Eq. (29)), which means that $\Lambda_1^f = f\Lambda_1$ endows U with a Poisson structure. Of course, Λ_1^f is compatible with Λ_0^f. Since Λ_0^f is nondegenerate on U, the pair $(\Lambda_0^f, \Lambda_1^f)$ possesses a recursion operator on U that is no other than the tensor field of type (1,1)

$$N^f = N - Y \otimes \frac{df}{f}$$

of $\mathcal{N}^f := (N^f, Y^f, \gamma^f, g^f)$. Then, (Λ_0^f, N^f) defines on U a symplectic Poisson–Nijenhuis structure.

Of course, the local model of (Λ_0^f, N^f) is known by Theorem 3.3. On the other hand, since $((\Lambda_0^f, E_0^f), \mathcal{N}^f)$, $\mathcal{N}^f := (N^f, Y^f, \gamma^f, g^f)$, is a Jacobi–Nijenhuis structure (see Proposition 2.11), its tensor fields verify Eqs. (19)–(22) and (25)–(27). Because $E_0^f = 0$ and Λ_0^f is nondegerate on U, from

$$N^f E_0^f = \Lambda_0^{f\#}(\gamma^f) + g^f E_0^f,$$

we get that $\gamma^f = 0$ on U. Then (cf. Proposition 2.11),

$$\gamma = -{}^{\mathrm{t}}N \frac{df}{f} + g^f \frac{df}{f}. \tag{133}$$

Taking into account this result, from

$${}^{\mathrm{t}}N^f(dg^f) = L_{Y^f}\gamma^f + g^f dg^f,$$

we deduce that g^f is a functional proper value of N^f or that g^f is constant on U. So, if s is a local coordinate system of M, centered at p, in which (Λ_0^f, N^f) has the expression of its model (cf. Theorem 3.3), then, we can easily deduce from this the local writings of Λ_0, $E_0 = -\Lambda_0^\#(df/f)$, N, Y and g in this system and, from Eq. (133), the one of γ.

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